Hypersemigroups Constructed on Especially Partially Ordered Sets

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Abstract: This paper links up the papers [3], [6] and [16]. It was proved that the hyperoperation $\circ$ introduced on partially-ordered sets in [16] is not associative for arbitrary underlying set and its ordering in full generality (see remark 7). At first the particular subsets with special orderings are defined and the operation $\circ$ modified. It is proved that the modified hyperoperation $\circ$ on such ordered subsets is associative. Finally a composed underlying set of hypergroupoid for which the multioperation is associative is constructed.

Key words: Binary hyperoperation, hypergroupoid, convex subset, meet and join-irreducible elements, whole, semiwhole and cramped complex, bunch.

1 Introduction

1.1 Definition A hypergroupoid (or a multigroupoid) is a pair $(M, \circ)$ where $M$ is a nonempty set and $\circ : M \times M \to P^*(M)$ is a binary hyperoperation called also a multioperation. $(P^*(M)$ is the system of all nonempty subsets of $M$).

A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $(a \circ b) \circ c = a \circ (b \circ c)$ for every triad $a, b, c \in M$.

1.2 Introduction We denote by $M$ a partially ordered set $M$ with the ordering $\leq$, with the greatest element $I$ and with the least element $O$ which will be inscribed in the next part of this article with $M = (M, \leq, I, O)$.

1.3 Definition We introduce for every element $u \in M$ a subset $U \subseteq M$ as follows: $U = \{u_i \mid u_i \geq u\}$ and we define on $M = (M, \leq, I)$ for arbitrary $x, y, z \in M$ the modified binary hyperoperation $\circ$ as follows:

$x \circ y = x \circ y = \{\min(X \cap Y)\}$. If $x \in y \circ z$ we put $x \circ (y \circ z) = (x \circ y) \circ z = y \circ z$

We inscribe then the set $M$ with such defined binary operation with $M = (M, \leq, \circ, I)$.

1.4 Definition - Remark We introduce the following very important concept. A subset $Di$ of $M = (M, \leq, \circ, I)$ is called dual ideal of $M$ if $Di$ satisfies the following condition:

For $x, y \in Di$ the relation $x \circ y \subseteq Di$ holds.
The subset $U$ defined in 1.3 is the dual ideal of the element $u \in \mathcal{M} = (M \leq, \diamond, I)$.

1.5 Remark We recall the basic notions: Let $M$ be a finite partly ordered set. The hyperoperation of multiplication $\diamond$ on $(\mathcal{M} = (M \leq, \diamond, I))$ is idempotent and commutative. Every upper-ideal of $\mathcal{M} = (M, \leq, \diamond, I)$ is identical with the dual-ideal of commutative hypergroupoid.

1.6 Remark The binary hyperoperation $\diamond$ defined on partly ordered sets is not associative for arbitrary orderind of the carrier set. It is obvious from the following example.

![Figure 1](image)

We show in this example that the hyperoperation $\diamond$ given on ordered set on Figure 1 is not associative. We prove that $(x \diamond y) \diamond z \neq x \diamond (y \diamond z)$. We denote $x \diamond y$ as $W$. From the definition of the hyperoperation $\diamond$ the set $W = \{w_1, w_2, w_3\}$. 

$W \diamond z = \{w_1 \diamond z\} \cup \{w_2 \diamond z\} \cup \{w_1 \diamond z\} = \{a\} \cup \{a\} \cup \{w_3\} = \{a, w_3\}$.

Similarly we denote $y \diamond z$ as $U$. Then $U = \{w_3\}$ and $x \diamond U = x \diamond w_3 = \{w_3\}$. Hence we proved Remark 1.6.

1.7 Definition Let $a$ and $b$ be a pair of elements of the finite partly ordered set $M$ such that $a \leq b$. We define as the interval bounded by the elements $a$ and $b$ and denote $[a, b]$ the set of all the elements $x$ of $M$ for which $a \leq x \leq b$ holds.

1.8 Definition A subset $P$ of finite partly ordered set $M$ will be said to be convex if it contains with any pair of elements $a, b$ ($a \leq b$) every element $x$ of the interval $[a, b]$; i.e. if $a, b \in P$ and $x \in [a, b]$ imply $x \in P$. (For $a = b$ the interval $[a, b]$ is onepoint subset).

1.9 Definition An element $x$ of finite partly ordered set $M$ different from the least one is called join-irreducible iff there exists one and only one element $y \in M$ which is previous element of $x$. We denote this fact by $y \rightarrow x$. Similarly an element $x \in M$
different from the greatest one is called meet-irreducible if there exits one and only one element $y \in M$ for which $x \rightarrow y$.

**1.10 Definition** By the length of a chain consisting of $r + 1$ elements that is of the form

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_r$$

(where the notation $x_i \rightarrow x_{i+1}$ means that the element $x_i$ is covered by the element $x_{i+1}$ - see [12]) we mean the non-negative number $r$. We assign to every element $x \in M$ the length of its chain from the greatest element. We denote this distance by $d(x)$.

**1.11 Definition** Let $M$ be a finite partly ordered set. A subset $P \subseteq M$ with the greatest meet-irreducible element $b \in M$ and the least join-irreducible element $a \in M$ or the minimal elements $a_i, i = 1, 2, \ldots, n$ is said whole complex where for every pair $x, y \in P$ for which $d(x) > d(y)$ we have $x \mathord{\not\mathord{<}} y$ and for $d(x) = d(y)$ we have $x \parallel y$.

**1.12 Definition** Let $M$ be a finite partly ordered set. A subset $Q \subseteq M$ with the greatest meet-irreducible element $b \in M$ and least join-irreducible element $a \in M$ or the minimal elements $a_i \in M$ for which every interval $[a_i, b], i = 1, 2, \ldots, n$ contains all elements $x \in M$ for which $a_i < x < b$ and the elements $u, v \in Q, u \in [a_i, b], v \in [a_j, b], i \neq j$ are incomparable is said semi-whole complex.

![Diagram of whole and semi-whole complexes](image.png)

Whole complex $P$  
Semi-whole complex $P$  

*Figure 2*
1.13 Lemma Let $M$ be a finite partly ordered set which is constructed as a union of whole and semi-whole complexes. The individual components are in the set $M$ ordered such that the join-irreducible elements $b_i$ and meet-irreducible elements $a_i$ are connected by chains $[a_i, b_i]$ of the length one. Then $M = (M \leq, \diamond, I)$ is a semihypergroup.

Proof.

1) At first we prove the associativity of the modified operation $\diamond$ on any whole complex $P \subseteq M$. Let $x, y, z$ be arbitrary elements of $P$. We denote $y \diamond z = A \subseteq P$.

a) At first we suppose that $x \succ v$ for every $v$ from the subset $A$. (the subset $A$ may be equal to $y$ or $z$.) For both situations $x \diamond A = x$. Now we prove that also $(x \diamond y) \diamond z = x$. It is obvious that also $x \succ y$ and hence $x \diamond y = x$. Similarly $x \succ z$ which implies $x \diamond z = x$. We have $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

b) Now let us contrary suppose that $x \prec v$ for every $v$ from the subset $A \subseteq P$. It is obvious that $y \preceq v$. Let $x \preceq v$ then $x \diamond y = y$ and $y \diamond z = A$. Now let $x \succ y$. Simultaneously the element $x$ underlies sub all elements $v$ from $A$ and from the relation $y \odot z$ follows also

$x \diamond y = x \diamond z = A$. We have proved the equality relation $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ again.

c) At last it remains the situation when $x \parallel v$ for every $v \in A$. This situation is possible only when $x \in A$ and from the definition of the modified hyperoperation $\diamond$ follows $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

2) Now we suppose that all three elements are not in the same complex.

a) At first we suppose that all three elements are in three different whole complexes $P_1, P_2, P_3 \subseteq M$. Let $x \in P_1, y \in P_2, z \in P_3$ and complexes are linear comparable such that $P_2$ is over $P_3$ and $P_1$ over $P_2$. Then $y \diamond z = y$ hence $y \odot z = y$ and $x \diamond y = x$. From the opposite side $x \diamond y = x$ and $x \diamond z = x$. We have $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

b) Now we suppose, that two of complexes are comparable and the third is with this both uncomparable. Without loss of generality let $P_2$ and $P_3$ are comparable and such that $P_2$ is over $P_3$. Then $y \odot z = y$. Evidently $x \diamond y$ is equal to any join-irreducible element lying in the smallest complex which is over $P_1$ and $P_2$. We denote it by $w$. Simultaneously from $x \diamond y = w$ and $y \diamond y = y$ follows $(x \diamond y) \odot z = w \diamond z = w$ and the associativity is proved again.

c) At least we suppose that all three complexes are incomparable. Now the relations $y \odot z \parallel x$ and $x \diamond y \parallel z$ hold. From the definition of dual ideal (1.4) $x \in D_i(x)$ and $y \odot z \subseteq D_i(y \odot z)$ Hence $\min(D_i(x) \cap D_i(y \odot z)) = \min(D_i(x \odot y \odot z)) = x \diamond (y \odot z)$ and similarly $\min(D_i(x \odot y) \cap D_i(z) = \min(D_i(x \odot y \odot z)) = (x \odot y) \odot z$ and the associativity of the operation $\diamond$ on finite partly ordered set which is constructed as a union of whole and semi-whole complexes is proved.

1.14 Definition Let $M$ be a finite partly ordered set. Further we suppose that the element $b \in M$ is a meet-irreducible element and $a_i \in M, i = 1, 2, ..., n$ are incomparable minimal or join-irreducible elements, for which $a_i < b$. We assign all whole complexes $[a_i, b]$ by $P_i$. The union $P = \bigcup_{i \in I} P_i$ will be called a branch.

1.15 Lemma Let $M$ be a finite partly ordered, $P \subseteq M$ a bunch. $P = (P \leq, \diamond, b)$ is a semihypergroup.
Proof.

1) $P = \bigcup_{i \in I} P_i$. Let us suppose that $x, y, z \in P_i$. The correctness of the relation $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ follows from Lemma 1.13.

2) Let $(P_i - b) \cap (P_j - b) = \emptyset$ and $y, z \in P_i$, $x \in P_j$. Then $y \diamond z \in P_i$ and $x \diamond (y \diamond z) = b$ which is the smallest element over both expressions $y \diamond z$ and $x$. Simultaneously $x \diamond y = b$ and also $(x \diamond y) \diamond z = b$ for $z < b$ and the associativity is satisfied.

3) Let $(P_i - b) \cap (P_j - b) \neq \emptyset$. If $x, y, z \in P_i \cap P_j$ then the associativity of the operation $\diamond$ follows from Lemma 1.14.

4) Let $y, z \in P_i$ and $x \in P_j$. Simultaneously let $x \neq b \wedge y \neq b \wedge z \neq b$.

   a) Let $y \diamond z = A \in P_i$ and $A \not\subseteq P_j$. We suppose that every element from $P_j - P_i$ is covered only by one element from $P_i$. (See Figure 3). Then $x \diamond A = x \diamond (y \diamond z) = v$. Hence $x \diamond y = v$ for $v$ is the smallest element lying over both elements $x$ and $y$. Simultaneously $z < v$ and we have $x \diamond (y \diamond z) = (x \diamond y) \diamond y$.

   b) Let $y \diamond z = A \in P_i$ and $A \not\subseteq P_j$. We suppose that every element from $P_j - P_i$ is covered by the set $C$ of elements from $P_i$. Then $x \diamond A = x \diamond (y \diamond z) = C$. Hence $x \diamond y \in C$ for $C$ is the set of smallest elements lying over both elements $x$ and $y$. Simultaneously $z < v$ for every element $v \in C$ and we have $x \diamond (y \diamond z) = (x \diamond y) \diamond y$. The same situation rises when $v \in A$ or $C \subseteq A$.

   c) Finally let $x$ is less then all elements of $y \diamond z = A$. Then $x \diamond (y \diamond z) = A$. On contrary let $x < y$ or $x < z$. We will suppose without loss of generality that $x < y$ then $x \diamond y = y$ and $y \diamond z = A$. Hence $x \diamond (y \diamond z) = (x \diamond y) \diamond y$ again and the Lemma is proved.

Figure 3

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1.16 Theorem. Let $M$ be a finite partly ordered set which is constructed as a union of whole, semi-whole complexes and bunches. Then $\mathcal{M} = (M \leq, \diamond, I)$ is a hypersemigroup. We call such hypersemigroup composed hypersemigroup for short $C$-hypersemigroup.

Proof. The affirmation follows from the Lemma 1.13 and Lemma 1.15.

References


