Hypersemigroups Constructed on Especially Partially Ordered Sets

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Abstract: This paper links up the papers [3], [6] and [16]. It was proved that the hyperoperation $\circ$ introduced on partially-ordered sets in [16] is not associative for arbitrary underlying set and its ordering in full generality (see remark 7). At first the particular subsets with special orderings are defined and the operation $\circ$ modified. It is proved that the modified hyperoperation $\diamond$ on such ordered subsets is associative. Finally a composed underlying set of hypergroupoid for wich the multioperation is associative is constructed.

Key words. Binary hyperoperation, hypergroupoid, convex subset, meet and join-irreducible elements, whole, semiwhole and cramped complex, bunch,

1 Introduction

1.1 Definition *A hypergroupoid* (or *a multigroupoid*) is a pair $(M, \circ)$ where M is a nonempty set and $\circ : M \times M \to \mathcal{P}^*(M)$ is a binary hyperoperation called also a multioperation. ($\mathcal{P}^*(M)$ is the system of all nonempty subsets of M).
A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $(a \circ b) \circ c = a \circ (b \circ c)$ for every triad a, b, $c \in M$.

1.2 Introduction We denote by $\mathcal{M}$ a partially ordered set M with the ordering $\leq$, with the greatest element I and with the least element O which will be inscribed in the next part of this article with $\mathcal{M} = (M, \leq, I, O)$

1.3 Definition We introduce for every element $u \in M$ a subset $U \subseteq M$ as follows: $U = \{u_i \mid u_i \geq u\}$ and we define on $\mathcal{M} = (M, \leq, I)$ for arbitrary $x, y, z \in M$ the modified binary hyperoperation $\diamond$ as follows:
$x \diamond y = x \circ y = \{min(X \cap Y)\}$. If $x \in y \circ z$ we put $x \diamond (y \diamond z) = (x \diamond y) \diamond z = y \diamond z$
We inscribe then the set $\mathcal{M}$ with such defined binary operation with $\mathcal{M} = (M \leq, \diamond, I)$.

1.4 Definition - Remark We introduce the following very important concept. A subset Di of $\mathcal{M} = (M \leq, \diamond, I)$ is called *dual ideal* of $\mathcal{M}$ if Di satisfies the following condition:

For $x, y \in Di$ the relation $x \diamond y \subset Di$ holds.

The subset U defined in 1.3 is the dual ideal of the element $u \in \mathcal{M} = (\,M \leq, \diamond, I)$.

1.5 Remark We recall the basic notions: Let M be a finite partly ordered set. The hyperoperation of multiplication $\diamond$ on $(\mathcal{M} = (\,M \leq, \diamond, I)$ is idempotent and commutative. Every upper-ideal of $\mathcal{M} = (\,M, \leq, \diamond, I)$ is identical with the dual-ideal of commutative hypergroupoid.

1.6 Remark The binary hyperoperation $\diamond$ defined on partly ordered sets is not associative for arbitrary orderind of the carrier set. It is obvious from the following example.

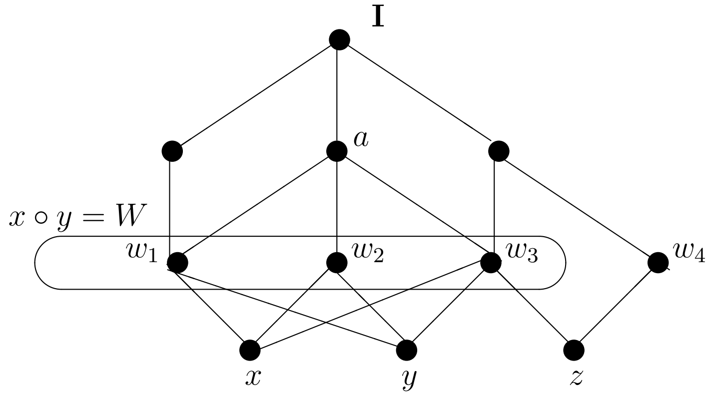

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Figure 1

We show in this example that the hyperoperation $\diamond$ given on ordered set on Figure 1 is not associative. We prove that $(x \diamond y) \diamond z \neq x \diamond (y \diamond z)$. We denote $x \diamond y$ as W. From the definition of the hyperoperation $\circ$ the set $W = \{w_1,\, w_2,\, w_3\}$.

$$W \diamond z = \{w_1 \diamond z\} \cup \{w_2 \diamond z\} \cup \{w_1 \diamond z\} = \{a\} \cup \{a\} \cup \{w_3\} = \{a,\, w_3\}.$$

Similarly we denote $y \diamond z$ as U. Then $U = \{w_3\}$ and $x \diamond U = x \diamond w_3 = \{w_3\}$. Hence we proved Remark 1.6.

1.7 Definition Let a and b be a pair of elements of the finite partly ordered set M such that $a \leq b$. We define as the interval bounded by the elements a and b and denote $[a, b]$ the set of all the elements x of M for which $a \leq x \leq b$ holds.

1.8 Definition A subset P of finite partly ordered set M will be said to be convex if it contains with any pair of elements a, b $(a \leq b)$ every element x of the interval $[a, b]$; i.e. if $a, b \in P$ and $x \in [a, b]$ imply $x \in P$. (For $a = b$ the interval $[a, b]$ is onepoint subset).

1.9 Definition An element x of finite partly ordered set M different from the least one is called join-irreducible iff there exists one and only one element $y \in M$ which is previous element of x. We denote this fact by $y \multimap\!\!\prec x$. Similarly an element $x \in M$

different from the greatest one is called meet-irreducible if there exits one and only one element $y \in M$ for which $x \prec y$

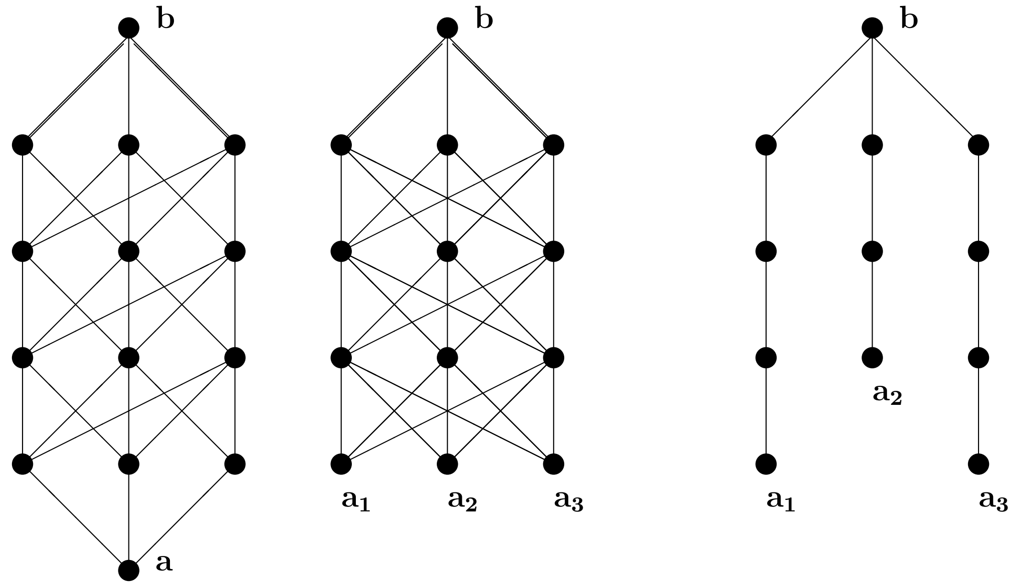
1.10 Definition By the length of a chain consisting of $r+1$ elements that is of the form

$$x_0 \prec x_1 \prec x_2 \prec \ . \ . \ . \prec x_r \qquad [x_0, \ x_r]$$

(where the notation $x_i \prec x_{i+1}$ means that the element x_i is covered by the element x_{i+1} - see [12]) we mean the non-negative number r. We assign to every element $x \in M$ the length of its chain from the greatest element. We denote this distance by $d(x)$.

1.11 Definition Let M be a finite partly ordered set. A subset $P \subseteq M$ with the greatest meet-irreducible element $b \in M$ and the least join-irreducible element $a \in M$ or the minimal elements $a_i, i = 1, 2, ..., n$ is said whole complex where for every pair $x, y \in P$ for which $d(x) > d(y)$ $x < y$ and for $d(x) = d(y)$ $x \parallel y$.

1.12 Definition Let M be a finite partly ordered set. A subset $Q \subseteq M$ with the greatest meet-irreducible element $b \in M$ and least join-irreducible element $a \in M$ or the minimal elements $a_i \in M$ for which every interval $[a_i, b]$, $i = 1, 2, ..., n$ contains all elements $x \in M$ for which $a_i < x < b$ and the elements $u, v \in Q, u \in [a_i, b], v \in [a_j, b], i \neq j$ are incomparable is said semi-whole complex.



Whole complex P

Semi-whole complex P

Figure 2

1.13 Lemma Let M be a finite partly ordered set which is constructed as a union of whole and semi-whole complexes. The individual components are in the set M ordered such that the join-irreducible elements b_i and meet-irreducible elements a_i are connected by chains $[a_i, b_i]$ of the length one. Then $\mathcal{M} = (\ M \leq, \diamond, I)$ is a semihypergroup.

Proof.

1) At first we prove the associativity of the modified operation $\diamond$ on any whole complex $P \subseteq M$. Let x, y, z be arbitrary elements of P. We denote $y \diamond z = A \subseteq P$.

a) At first we suppose that $x > v$ for every v from the subset A. (the subset A may be equal to y or z.) For both situations $x \circ A = x$. Now we prove that also $(x \circ y) \circ z = x$. It is obvious that also $x > y$ and hence $x \diamond y = x$. Similarly $x > z$ which implies $x \diamond z = x$. We have $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

b) Now let us contrary suppose that $x < v$ for every v from the subset $A \subseteq P$. It is obvious that $y \leq v$. Let $x \leq y$ then $x \diamond y = y$ and $y \diamond z = A$. Now let $x > y$. Simultaneously the element x underlies sub all elements v from A and from the relation $y \diamond z$ follows also $x \diamond y = x \diamond z = A$. We have proved the equality relation $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ again.

c) At last it remains the situation when $x \parallel v$ for every $v \in A$. This situation is possible only when $x \in A$ and from the definition of the modified hyperoperation $\diamond$ follows $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

2) Now we suppose that all three elements are not in the same complex.

a) At first we suppose that all three elements are in three different whole complexes $P_1, P_2, P_3 \subset M$ Let $x \in P_1, y \in P_2, z \in P_3$ and complexes are linear comparable such that P_2 is over P_3 and P_1 over P_2. Then $y \diamond z = y$ hence $y \diamond z = y$ and $x \diamond y = x$. From the opposide side $x \diamond y = x$ and $x \diamond z = x$. We have $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

b) Now we suppose, that two of complexes are comparable and the third is with this both uncomparable. Without loss of generality let P_2 and P_3 are conparable and such that P_2 is over P_3. Then $y \diamond z = y$. Evidently $x \circ y$ is equal to any join-irreducible element lying in the smallest complex which is over P_1 and P_2. We denote it by w. Simultaneously from $x \diamond y = w$ and $y \diamond y = y$ follows $(x \diamond y) \diamond z = w \diamond z = w$ and the associativity is proved again.

c) At least we suppose that all three complexes are incomparable. Now the relations $y \diamond z \parallel x$ and $x \diamond y \parallel z$ hold. From the definition of dual ideal (1.4) $x \in D_i(x)$ and $y \diamond z \subset D_i(y \diamond z)$ Hence $\min(D_i(x) \cap D_i(y \diamond z)) = \min(D_i(x \diamond y \diamond z)) = x \diamond (y \diamond z)$ and similarly $\min(D_i(x \diamond y) \cap D_i(z) = \min(D_i(x \diamond y \diamond z)) = (x \diamond y) \diamond z$ and the associativity of the operation $\diamond$ on finite partly ordered set which is constructed as a union of whole and semi-whole complexes is proved.

1.14 Definition Let M be a finite partly ordered set. Further we suppose that the element $b \in M$ is a meet-irreducible element and $a_i \in M, i = 1, 2, ..., n$ are incomparable minimal or join-irreducible elements, for which $a < b$. We assign all whole complexes $[a_i, b]$ by P_i. The union $P = \bigcup_{i \in I} P_i$ will be called a branch.

1.15 Lemma Let M be a finite partly ordered, $P \subseteq M$ a bunch. $\mathcal{P} = (\ P \leq, \diamond, b)$ is a semihypergroup.

Proof.

1) $P = \bigcup_{i \in I} P_i$. Let us suppose that $x, y, z \in P_i$. The correctness of the relation $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ follows from Lemma 1.13.

2) Let $(P_i - b) \cap (P_j - b) = \emptyset$ and $y, z \in P_i$, $x \in P_j$. Then $y \diamond z \in P_i$ and $x \diamond (y \diamond z) = b$ which is the smallest element over both expressions $y \diamond z$ and x. Simultaneously $x \diamond y = b$ and also $(x \diamond y) \diamond z = b$ for $z < b$ and the associativity is satisfied.

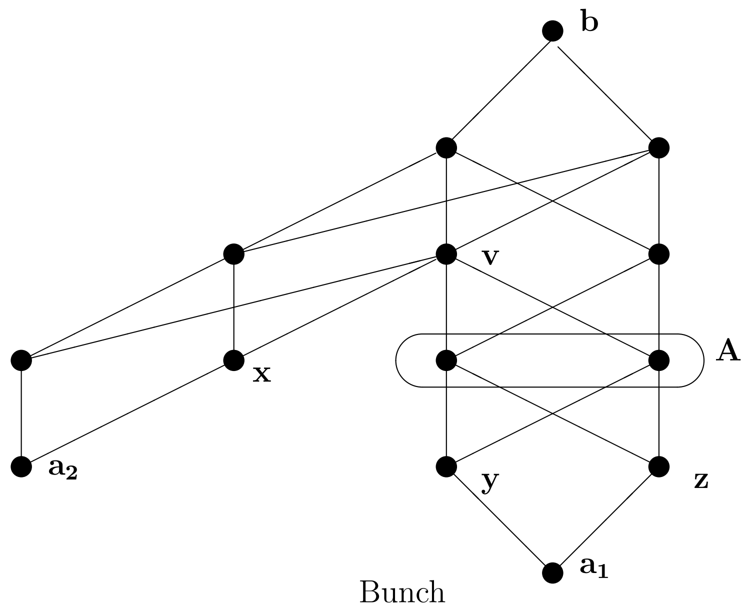
3) Let $(P_i - b) \cap (P_j - b) \neq \emptyset$. If $x, y, z \in P_i \cap P_j$ then the associativity of the operation $\diamond$ follows from Lemma 1.14.

4) Let y, z lies in P_i and $x \in P_j$. Simultaneously let $x \neq b \wedge y \neq b \wedge z \neq b$.

a) Let $y \diamond z = A \in P_i$ and $A \not\subset P_j$. We suppose that every element from $Pj - P_i$ is covered only by one element from P_i. (See Figure 3). Then $x \diamond A = x \diamond (y \diamond z) = v$. Hence $x \diamond y = v$ for v is the smallest element lying over both elements x and y. Simultaneously $z < v$ and we have $x \diamond (y \diamond z) = (x \diamond y) \diamond y$.

b) Let $y \diamond z = A \in P_i$ and $A \not\subset P_j$. We suppose that every element from $Pj - P_i$ is covered by the set C of elements from P_i. Then $x \diamond A = x \diamond (y \diamond z) = C$. Hence $x \diamond y \in C$ for C is the set of smallest elements lying over both elements x and y. Simultaneously $z < v$ for every element $v \in C$ and we have $x \diamond (y \diamond z) = (x \diamond y) \diamond y$. The same situation rises when $v \in A$ or $C \subset A$.

c) Finally let x is less then all elements of $y \diamond z = A$. Then $x \diamond (y \diamond z) = A$. On contrary let $x < y$ or $x < z$ We will suppose without loss of generality that $x < y$ then $x \diamond y = y$ and $y \diamond z = A$ Hence $x \diamond (y \diamond z) = (x \diamond y) \diamond y$ again and the Lemma is proved.



Bunch

Figure 3

1.16 Theorem Let M be a finite partly ordered set which is constructed as a union of whole, semi-whole complexes and bunches. Then $\mathcal{M} = (\ M \leq, \diamond, I)$ is a hypersemigroup. We call such hypersemigroup composed hypersemigroup for short C-hypersemigroup.

Proof. The affirmation follows from the Lemma 1.13 and Lemma 1.15.

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