Existence of solutions of fractional order functional differential equations with infinity delay

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Fractional differential equations have been of great interest recently. In cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, etc. for details, see the monographs of Miller and Ross[3], Podlubny[4], Samko[6] and papers Benchohra and Henderson[1], Yu and Gao[7], Podlubny and and Petráš[5].

In this paper we investigate fractional order functional differential equations with infinite delay of the form

$$D^{\alpha}y(t) = f(t, y_t), \quad t \in J = [0, b], \quad 0 < \alpha < 1, \tag{1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2}$$

where D^{α} is the standart Riemann-Liouville fractional derivative, $f : J \times B \to R$, $\phi \in B$, $\phi(0) = 0$, B is a phase space. For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of B defined

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

The fractional integral of order $\alpha > 0$ of a function $h : R^+ \to R$ is defined by

$$I_0^{\alpha}h(t) = \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)} h(s)ds.$$

The fractional derivative of order $\alpha > 0$ of a continuous function $h : R^+ \to R$ is defined by

$$\frac{d^{\alpha}h(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} h(s)ds = \frac{d}{dt} I_a^{1-\alpha} h(t).$$

In the following we will assume that the space $(B, ||.||_B)$ is a seminormed linear functions mapping $(-\infty, 0]$ into R, and satisfying the following fundamental axioms which were introduced by Hale and Kato[2]:

(i) If $y : (-\infty, b] \to R$, and $y_0 \in B$, then for every $t \in [0, b]$ the following conditions hold:

$y_t \in B$,
$||y_t||_B \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)||y_0||_B$,
$|y(t)| \leq H||y_t||_B$, where $H \geq 0$ is a constant, $K : [0, b] \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M are independent of $y(.)$.

(ii) For the function $y(.)$ in (i), y_t is a B-valued continuous function on $[0, b]$.

(iii) The space B is complete.

Theorem 1 *Assume there exists $L > 0$ such that*

$$|f(t,u) - f(t,v)| \leq L||u - v||_B \quad \textit{for } t \in J \textit{ and every } u, v \in B.$$

If

$$\frac{b^\alpha K_b L}{\Gamma(\alpha+1)} < 1,$$

where $K_b = \sup\{|K(t)| : t \in [0,b]\}$, then there exists a unique solution for (1),(2) *on the interval* $(-\infty, b]$.

Acknowledgement

This research has been supported by the Czech Ministry of Education in the frames of MSM0021 60503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies and MSM0021630529 Research Intention Inteligent Systems in Automatization.

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