TOPOLOGY AND CAUSALITY STRUCTURES ON MINKOWSKI SPACE

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Abstract. We construct an example of a causal site (in the sense of Crane and Christensen) on a compact subspace of the Minkowski space. In the second part of the paper we show that it is possible to reconstruct the original topology of the Minkowski space from the previously constructed causality structure.

1. Preliminaries and introduction

Recall that a causal site [1] $(S, \sqsubseteq, \prec)$ is a set S of *regions* equipped with two binary relations $\sqsubseteq$, $\prec$, where $(S, \sqsubseteq)$ is a partial order having the binary suprema $\sqcup$ and the least element $\bot \in S$, and $(S \smallsetminus \{\bot\}, \prec)$ is a strict partial order (i.e. antireflexive and transitive), linked together by the following axioms, which are satisfied for all regions $a, b, c \in S$:

(i) $a \sqsubseteq b$ and $a \prec c$ implies $b \prec c$,
(ii) $b \sqsubseteq a$ and $c \prec a$ implies $c \prec b$,
(iii) $a \prec c$ and $b \prec c$ implies $a \sqcup b \prec c$.
(iv) There exits $b_a \in S$, called *cutting of a by b*, such that
  (1) $b_a \prec a$ and $b_a \sqsubseteq b$;
  (2) if $c \in S$, $c \prec a$ and $c \sqsubseteq b$ then $c \sqsubseteq b_a$.

Definition 1.1. *Let P be a (finite) set, $\pi \subseteq 2^P$. We say that (P, π) is a framework. The elements of P we call places, the set π we call framology.*

Definition 1.2. *Let (P, π) be a framework. Denote $P^d = \pi$ and $\pi^d = \{\pi(x) |\, x \in P\}$, where $\pi(x) = \{U |\, U \in \pi, x \in U\}$. Then (P^d, π^d) is the dual framework of (P, π). The places of the dual framework (P^d, π^d) we call abstract points or simply points of the original framework (P, π).*

2. The causal site on Minkowski space

Let $\mathbb{M} = \mathbb{R}^4$ be the Minkowski space. Then $\mathbb{M}$ has the structure of the 4-dimensional real vector space equipped with the bilinear form $\eta : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$, called Minkowski inner product. We define $p \leqslant q$ if the vector $q - p$ is

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non-past-oriented and non-spacelike, that is, if its time coordinate is non-negative and $\eta(q-p, q-p) \geq 0$. Now, we denote

$$\lozenge(p,q) = \{x \mid x \in \mathbb{M}, \eta(x-p, x-p) \geq 0, \eta(q-x, q-x) \geq 0\},$$

where $p, q \in \mathbb{M}$, $p \leqslant q$. Let us construct a causal site which reflects causality and topological properties of the Minkowski space $\mathbb{M}$. Denote

$$\mathbb{D} = \lozenge((1,0,0,0), (-1,0,0,0)).$$

Let D be the set of all nonempty intersections $\lozenge(p,q) \cap \mathbb{D}$ with $p, q \in \mathbb{Q}^4$, $p \leqslant q$. Now, let $(P, \cup, \cap)$ be the set lattice generated by the elements of D. Let $A, B \in P$ non-empty. We put $A \prec B$ if $A \neq B$ and for every $a \in A$, $b \in B$, $a \leqslant b$.

Proposition 2.1. *$(P, \subseteq, \prec)$ is a causal site.*

3. Reconstruction of topology

Consider a general causal site $(P, \sqsubseteq, \prec)$ and let us define appropriate framework structure on P. We say that a subset $F \subseteq P$ set is centered, if for every $x_1, x_2, \ldots, x_k \in F$ there exists $y \in P$, $y \neq \perp$ satisfying $y \sqsubseteq x_i$ for every $i = 1, 2, \ldots, k$. Let π be the family of all maximal centered subsets of P. Now, consider the framework (P, π) and its dual (P^d, π^d). Let (X, τ) be the topological space with $X = P^d = \pi$ and the topology τ generated by its closed subbase (that is, a subbase for the closed sets) π^d.

Theorem 3.1. *The topological space (X, τ), corresponding to the framework (P^d, π^d) and the causal site $(P, \sqsubseteq, \prec)$, is compact T_1.*

In the following theorem, let π be the family of all maximal centred subsets of P, where $(P, \subseteq, \prec)$ is the causal site constructed in the previous section. Using the framework duality, we may get back the original topology on $\mathbb{D}$.

Theorem 3.2. *The topological space (X, τ) corresponding to the framework (P^d, π^d) is homeomorphic to $\mathbb{D}$ equipped with the Euclidean topology.*

Note that more information the reader can find in the electronic version of the conference proceedings on the enclosed CD.

References

1. Christensen, J. D., Crane L., *Causal Sites as Quantum Geometry*, preprint, 1-21.
2. Engelking, R., *General topology*, Heldermann Verlag, Berlin (1989), pp. 1-532.
3. Ganter, B., Wille, R., *Formal Concept Analysis*, Springer-Verlag, Berlin (1999), pp.1-285.
4. Gratzer, G., *General Lattice Theory*, Birkhauser Verlag, Berlin (2003), pp. 1-663.
5. Zeeman, E.C., *Causality implies the Lorentz group*, J.Math.Phys. v.5 (1964), no.4, 490-493.

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