RECONSTRUCTION OF TOPOLOGY FROM CAUSALITY STRUCTURE OF MINKOWSKI SPACE

MARTIN MARIA KOVÁR

Abstract. The notion of causal site was introduced by J. D. Christensen and L. Crane. However, the known examples of causal sites were not able to capture all topological aspects of the space-time completely. In this paper we construct an example of a causal site (in the sense of Crane and Christensen) on a compact subspace of the Minkowski space, which is more suitable for capturing the topological relationships. In the second part of the paper we show that it is possible to reconstruct the original topology of the Minkowski space just only from the previously constructed causality structure.

1. Preliminaries and introduction

In this paper we will use two, perhaps less usual and rather less familiar algebraic structures. One of them is the notion of *causal site*, defined by J. D. Christensen and L. Crane in [1]. A causal site $(S, \sqsubseteq, \prec)$ is a set S of *regions* equipped with two binary relations $\sqsubseteq$, $\prec$, where $(S, \sqsubseteq)$ is a partial order having the binary suprema $\sqcup$ and the least element $\perp \in S$, and $(S \smallsetminus \{\perp\}, \prec)$ is a strict partial order (i.e. antireflexive and transitive), linked together by the following axioms, which are satisfied for all regions $a, b, c \in S$:

(i) $a \sqsubseteq b$ and $a \prec c$ implies $b \prec c$,
(ii) $b \sqsubseteq a$ and $c \prec a$ implies $c \prec b$,
(iii) $a \prec c$ and $b \prec c$ implies $a \sqcup b \prec c$.
(iv) There exits $b_a \in S$, called *cutting of a by b*, such that
  (1) $b_a \prec a$ and $b_a \sqsubseteq b$;
  (2) if $c \in S$, $c \prec a$ and $c \sqsubseteq b$ then $c \sqsubseteq b_a$.

Note that the notion of causal site was introduced by J. D. Christensen and L. Crane as a generalization of rather more frequently used and little bit older notion *causal set* of R. Sorkin. In [1] there are constructed several examples of causal sites, in which it is demonstrated how they can capture the causality structure of the spacetime. However, the topology of the spacetime has to arise in some physically conditioned way, from some physically conditioned reason, although it has not been sufficiently investigated. One of the possible candidates for generating the topology of the spacetime, in some way, are the causality relationships. Thus it is very natural to ask,

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whether the original topology of the spacetime may be recostructed from the structure satisfying Christensen-Crane's axioms. In this paper we will show that it is really possible at least for a certain compact subspace of the Minkowski space.

In the nature or in the physical universe, what ever it is, there are probably no existing real points like in classical Euclidean geometry, or at least, we cannot be sure of that. Points, as a useful mathematical abstraction, are infinitesimally small and thus cannot be measured or detected by no physical way. But what we can be sure that really exists, are various locations, containing concrete physical objects. Int his paper we will call these locations *places*. Topological relationships on sets of places are captured by the algebraic structure which we call the *framework* structure. This structure was introduced by the author and referred at several occasions and mathematical conferences.

Definition 1.1. *Let P be a (finite) set, $\pi \subseteq 2^P$. We say that (P, π) is a framework. The elements of P we call places, the set π we call framology.*

Although every topological space is a framework by the definition, the philosophy, as well as the elementary interpretation of a framework is very different from the usual interpretation of a topological space. The elements of the framology are not primarily considered as neighborhoods of places, although this seems to be also very natural. If P contains all the places that can be observed, the framology π contains the list of observations of the fact that the "virtual observer", or some physical object that "really exists" can be present at some places simultaneously. Places primarily have no geometrical meaning, they are considered just as elements of some set equipped with the framework structure. But in our everyday experience, places are some, possibly overlapping (and not necessarily precisely geometrically determined) regions, containing various physical objects (bodies, particles, etc.).

Definition 1.2. *Let (P, π) and (S, σ) be frameworks. A mapping $f : P \to S$ satisfying $f(\pi) \subseteq \sigma$ we call a framework morphism.*

Definition 1.3. *Let (P, π) be a framework. Denote $P^d = \pi$ and $\pi^d = \{\pi(x)|\, x \in P\}$, where $\pi(x) = \{U|\, U \in \pi, x \in U\}$. Then (P^d, π^d) is the dual framework of (P, π). The places of the dual framework (P^d, π^d) we call abstract points or simply points of the original framework (P, π).*

Note that the framework duality is a simple but handy tool for switching between the classical point-set representation (like in topological spaces) and point-less representation, introduced by the framework theory. Also it should be noted that an alternative (and more complex, but perhaps less illuminating for topological relationships) language to the framework theory yields the formal concept analysis, founded by B. Ganter and R. Wille [3].

We say that a family Φ of sets has the finite intersection property or shortly the f.i.p., if every its finite subfamily has a non-empty intersection.

A topological space is said to be compact, if every its open cover admits of a finite subcover, or equivalently, if every family of closed sets with f.i.p has a non-empty intersection. In particular, we do not assume the Hausdorff separation axiom as a part of the definition of compactness. Well-known Alexander's subbase lemma [2] ensures that the family of all open or closed sets, respectively, can be replaced by its corresponding open or closed subbase, respectively.

2. The causal site on Minkowski space

Let $\mathbb{M} = \mathbb{R}^4$ be the Minkowski space. Then $\mathbb{M}$ has the structure of the 4-dimensional real vector space equipped with the bilinear form $\eta : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$, called Minkowski inner product. Minkowski modification of the inner product is not positively definite as the usual inner product, but in the standard basis it is represented by the diagonal matrix with the diagonal entries $(1, -1, -1, -1)$. A vector $v \in \mathbb{M}$ is called timelike, if $\eta(v, v) > 0$, lightlike or null if $\eta(v, v) = 0$ and spacelike, if $\eta(v, v) < 0$. Further, the vector v is said to be future-oriented, if its first coordinate, which represents the time, is positive. Similarly, v is past-oriented, if its first coordinate is negative. We write $v \ll w$ for $v, w \in \mathbb{M}$ if the vector $w - v$ is timelike and future-oriented. In [1] the sets of the form $D(p, q) = \{x|\, x \in \mathbb{M}, p \ll x \ll q\}$, called diamonds. They are used in some examples for the construction of certain causal sites. In this setting, diamonds are open sets in the Euclidean topology, bounded by two light cones at points $p, q \in \mathbb{M}$. It is not difficult to show that open diamonds form a base for the Euclidean topology on $\mathbb{M}$. However, for the purpose of reconstruction of topology from the causality structure it is more convenient to consider the closed variant of diamonds.

We define $p \leqslant q$ if the vector $q-p$ is non-past-oriented and non-spacelike, that is, if its time coordinate is non-negative and $\eta((q - p, q - p) \geq 0$. We also denote $\mathbf{0} = (0, 0, 0, 0)$. Now, we put

$$J^+(p) = \{x|\, x \in \mathbb{M}, p \leqslant x\}$$

and

$$J^-(p) = \{x|\, x \in \mathbb{M}, x \leqslant p\}.$$

Lemma 2.1. *The sets $J^+(\mathbf{0})$ and $J^-(\mathbf{0})$ are closed with respect to the operation $+$ of the vector space $(\mathbb{M}, +)$.*

The proof of the previous lemma can be based on the fact, that any vector $x \in \mathbb{M}$ can be decomposed into the sum of two vectors, say $x = s + t$, where s has the time coordinate equal to 0 and t has all space coordinates equal to 0. Then $x \in J^+(\mathbf{0})$ is equivalent to $\|t\| \geq \|s\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{M}$. It follows from the triangle inequality that this property is preserved by the operation $+$ on $\mathbb{M}$. The details of the proof we leave to the reader as an exercise. Also the proof of the next lemma is left to the reader because of its simplicity. It consists of a standard process of verifying reflexivity, antisymmetry and transitivity of the relation $\leqslant$. Only the case of transitivity is less trivial, and Lemma 2.1 is helpful.

Lemma 2.2. *The binary relation $\leqslant$ is a partial order on $\mathbb{M}$.*

Now, we denote

$$\Diamond(p,q) = \{x \mid x \in \mathbb{M}, \eta(x-p, x-p) \geq 0, \eta(q-x, q-x) \geq 0\},$$

where $p, q \in \mathbb{M}$, $p \leqslant q$. Let us construct a causal site which reflects causality and topological properties of the Minkowski space $\mathbb{M}$. For simplicity, we restrict our considerations to some bounded, compact part of $\mathbb{M}^4$. Since all physical phenomena which can be observed could happen perhaps at a long, but still finite distance, it should not be a substantial limitation or loss of generality. Denote

$$\mathbb{D} = \Diamond((1,0,0,0), (-1,0,0,0)).$$

Let D be the set of all nonempty intersections $\Diamond(p,q) \cap \mathbb{D}$ with $p, q \in \mathbb{Q}^4$, $p \leqslant q$. Now, let $(P, \cup, \cap)$ be the set lattice generated by the elements of D. Since P can be represented by lattice polynomials (see, e.g. [4]), every element of P can be expressed by unions and intersections of finitely many elements of D, it is compact and closed with respect to the Euclidean topology on $\mathbb{M}$.

Lemma 2.3. *The family P is a closed base for the Euclidean topology on $\mathbb{D}$.*

Proof. Let U be an open set with respect to the Euclidean topology $\tau_{\mathbb{D}}$ on $\mathbb{D}$, induced from $\mathbb{M}$. Take a point $x \in U$. The set $\mathbb{D} \smallsetminus U$ is compact. For every $y \in \mathbb{D} \smallsetminus U$ there exist $p_y, q_y \in \mathbb{Q}^4$, $p_y \leqslant q_y$, such that $y \in \operatorname{int} \Diamond(p_y, q_y)$, where the interior is considered with respect to the Euclidean topology on $\mathbb{M}$, and $x \notin \Diamond(p_y, q_y)$. Since $\mathbb{D} \smallsetminus U$ is compact, there exist $y_1, y_2, \ldots, y_k \in \mathbb{D} \smallsetminus U$ with

$$\mathbb{D} \smallsetminus U \subseteq \bigcup_{i=1}^{k} \operatorname{int} \Diamond(p_{y_i}, q_{y_i}).$$

Then

$$x \in \bigcap_{i=1}^{k} (\mathbb{D} \smallsetminus \Diamond(p_{y_i}, q_{y_i})) = \mathbb{D} \smallsetminus \bigcup_{i=1}^{k} (\Diamond(p_{y_i}, q_{y_i}) \cap \mathbb{D}) \subseteq U,$$

and the closed set $\bigcup_{i=1}^{k} (\Diamond(p_{y_i}, q_{y_i}) \cap \mathbb{D})$ is an element of P. Hence, every set U, which is open with respect to $\tau_{\mathbb{D}}$, is a union of complements of elements of P, which are closed in the same topology. Then P forms a closed base for $\tau_{\mathbb{D}}$. □

Let $A, B \in P$ non-empty. We put $A \prec B$ if $A \neq B$ and for every $a \in A$, $b \in B$, $a \leqslant b$.

Proposition 2.1. *$(P, \subseteq, \prec)$ is a causal site.*

Proof. First of all, we need to show that $\prec$ is a transitive on the set $P \setminus \{\varnothing\}$. Suppose that $A \prec B$ and $B \prec C$, where A, B, C are non-empty. Let $a \in A$, $c \in C$. Since $B \neq \varnothing$, there is some $b \in B$. The vectors $b-a$ and $c-b$ are non-spacelike and non-past-oriented. Then also the vector $c-a = (c-b)+(b-a)$ is also non-space-like and non-past-oriented. Suppose that $A = C$. Then $A \prec B$ and $B \prec A$. Taking any $a \in A$ and $b \in B$, we get that both vectors $a - b$ and $b - a$ are non-spacelike and non-past-oriented, which gives $a = b$. Then $A = B$ is a singleton, but this equality contradicts to the definition of the relation $\prec$. Thus $\prec$ is transitive.

Since $\subseteq$ is the set inclusion, the axioms (i)-(iii) are satisfied trivially. Let us check the axiom (iv). Let $A \in P$, $A \neq \varnothing$. Denote

$$O_A = \{p | \, p \in \mathbb{D}, A \subseteq J^+(p)\}.$$

At least, $\mathbf{0} \in O_A$, so $O_A \neq \varnothing$. Let $L \subseteq O_A$ be a non-empty linearly ordered chain. We will show that L has an upper bound in O_A. Consider the net $id_L(L, \leqslant)$. Since $\mathbb{D}$ is compact, $id_L(L, \leqslant)$ has a cluster point, say $p_L \in \mathbb{D}$. Suppose that there is some $l \in L$ such that $p_L \notin J^+(l)$. Since the set $J^+(l)$ is closed in $\mathbb{M}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(p_A) \cap J^+(l) = \varnothing$. By the definition of the cluster point, there exists $m \in L$, $l \leqslant m$, such that $m \in B_\varepsilon(p_A)$. Then $m \in J^+(m) \cap B_\varepsilon(p_A)$, but this is not possible since $J^+(m) \subseteq J^+(l)$. Hence, $p_L \in \bigcap_{l \in L} J^+(l)$, which means that p_L is an upper bound of L in $\mathbb{D}$. It remains to show that $A \subseteq J^+(p_A)$. Suppose conversely, that there exists some $r \in A \setminus J^+(p_A)$. Since $J^+(p_A)$ is closed in $\mathbb{M}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(r) \cap J^+(p_A) = \varnothing$. Since p_A is a cluster point of the net $id_L(L, \leqslant)$, there exists $n \in L$, $n \in B_{\varepsilon/2}(p_A)$. Then $r \in A \subseteq J^+(n)$. Denote $q = r + (p_A - n)$. The vector q is the translation of r by the vector $p_A - n$, and $J^+(p_A)$ is the translation of the cone $J^+(n)$ by the same vector, so $q \in J^+(p_A)$. Now, $0 < \varepsilon \leq \|r - q\| = \|n - p_A\| < \frac{\varepsilon}{2}$, which is a contradiction. Thus $A \subseteq J^+(p_A)$, and so $p_A \in O_A$ is the upper bound of the chain L. Let M_A be the set of all maximal elements of O_A (with respect to the order $\leqslant$). By Zorn's Lemma, for every $p \in O_A$ there exists $m \in M_A$ such that $p \leqslant m$. We put

$$A_\perp = \bigcup_{m \in M_A} J^-(m),$$

and for $B \in P$, $B \neq A$ we denote

$$B_A = B \cap A_\perp.$$

Let $b \in B_A$, $a \in A$. By the definition of B_A, there exists some $m \in M_A$ with $b \in J^-(m)$, so $b \leqslant m$. We also have $a \in A \subseteq J^+(m)$, so $m \leqslant a$. Then $b \leqslant a$, which implies $B_A \prec A$.

Suppose that $C \prec A$, $C \subseteq B$ for some $C \in P$. Let $c \in C$. If $a \in A$, then $c \leqslant a$, which gives $a \in J^+(c)$. Therefore, $A \subseteq J^+(c)$. Then $c \in O_A$, so there exists $m \in M_A$, such that $c \leqslant m$. Then $c \in J^-(m) \subseteq A_\perp$. Hence, $C \subseteq A_\perp$, which together with $C \subseteq B$ gives the requested inclusion $C \subseteq B_A$. $\square$

3. Reconstruction of topology

Now we will especially concentrate on the possibility of reconstruction of the original topology on $\mathbb{D}$ from the causality structure of $(P, \subseteq, \prec)$ that we have constructed in the previous section.

Consider a general causal site $(P, \sqsubseteq, \prec)$ and let us define appropriate framework structure on P. We say that a subset $F \subseteq P$ set is centered, if for every $x_1, x_2, \dots, x_k \in F$ there exists $y \in P$, $y \neq \bot$ satisfying $y \sqsubseteq x_i$ for every $i = 1, 2, \dots, k$. If $\mathcal{L} \subseteq 2^P$ is a chain of centered subsets of P linearly ordered by the set inclusion $\subseteq$, then $\bigcup \mathcal{L}$ is also a centered set. Then every centered $F \subseteq P$ is contained in some maximal centered $M \subseteq P$. Let π be the family of all maximal centered subsets of P. Now, consider the framework (P, π) and its dual (P^d, π^d). Let (X, τ) be the topological space with $X = P^d = \pi$ and the topology τ generated by its closed subbase (that is, a subbase for the closed sets) π^d.

Theorem 3.1. *The topological space (X, τ), corresponding to the framework (P^d, π^d) and the causal site $(P, \sqsubseteq, \prec)$, is compact T_1.*

Proof. By Alexander's subbase lemma, for proving the compactness of (X, τ) it is sufficient to show, that any subfamily of π^d having the f.i.p., has a nonempty intersection. The subbase for the closed sets of (X, τ) has the form $\pi^d = \{\pi(x) |\, x \in P\}$, so any subfamily of π^d can be indexed by a subset of P. Let $F \subseteq P$ and suppose that for every $x_1, x_2, \dots, x_k \in F$ we have

$$\pi(x_1) \cap \pi(x_2) \cap \dots \cap \pi(x_k) \neq \varnothing.$$

Then there exists $U \in \pi$ such that $U \in \pi(x_1) \cap \pi(x_2) \cap \dots \cap \pi(x_k)$, so $x_i \in U$ for every $i = 1, 2, \dots, k$. Since U is a (maximal) centered family, there exists $\bot \neq y \in P$ such that $y \sqsubseteq x_i$ for every $i = 1, 2, \dots, k$. Thus F is a centered family, contained in some maximal centered family $M \subseteq P$. But then we have $M \in \pi$, so

$$M \in \bigcap_{x \in M} \pi(x) \subseteq \bigcap_{x \in F} \pi(x) \neq \varnothing.$$

Hence, (X, τ) is compact.

Let $U, V \in X = \pi$, $U \neq V$. Since both are maximal centered subfamilies of P, none of them can contain the other one. So, there exist $x, y \in P$ such that $x \in U \smallsetminus V$ and $y \in V \smallsetminus U$. Then $U \in \pi(x)$, $V \notin \pi(x)$, $V \in \pi(y)$, $U \notin \pi(y)$. Thus $X \smallsetminus \pi(x)$, $X \smallsetminus \pi(y)$ are open sets in (X, τ) containing just one of the points U, V. So the topological space (X, τ) satisfies the T_1 axiom. □

In the following theorem, let π be the family of all maximal centred subsets of P, where $(P, \subseteq, \prec)$ is the causal site constructed in the previous section. Using the framework duality, we may get back the original topology on $\mathbb{D}$.

Theorem 3.2. *The topological space (X, τ) corresponding to the framework (P^d, π^d) is homeomorphic to $\mathbb{D}$ equipped with the Euclidean topology.*

Proof. As we already defined before $X = P^d = \pi$. Note that any point $p \in \mathbb{D}$ defines a maximal centred subset of P, say $f(p) = \{C | \, C \in P, p \in C\}$. The family $f(p)$ obviously is centered, since P is closed under finite intersections and $f(p)$ contains those elements of P, whose contain p. Let Q be another centered family such that $f(p) \subseteq Q \subseteq P$. Suppose that there is some $F \in Q$, such that $p \notin F$. The set $\mathbb{D} \smallsetminus F$ is open with respect to the Euclidean topology, induced from $\mathbb{M}$ on $\mathbb{D}$, so there exist $u, v \in \mathbb{Q}^4$, $u \leqslant v$, such that $p \in \Diamond(u,v) \subseteq \mathbb{D} \smallsetminus F$. But $\Diamond(u,v) \in P$, so $\Diamond(u,v) \in f(p) \subseteq Q$, while $\Diamond(u,v) \cap F = \varnothing$. This contradicts to the assumption that Q is centered. Thus all elements of Q contain p, which means that $Q = f(p)$. Now it is clear that $f(p)$ is a maximal centred subfamily of P.

Conversely, a maximal centered subfamily $Q \in \pi$ has a nonempty intersection, because of compactness of $\mathbb{D}$. If $\{x, y\} \subseteq \bigcap_{F \in Q} F$, where $x \neq y$, then there exist $u, v \in \mathbb{Q}^4$, $u \leqslant v$, such that $x \in \Diamond(u,v)$ and $y \notin \Diamond(u,v)$. Then $Q \cup \{\Diamond(u,v)\} \subseteq P$ is an extension of Q which is also centered, which contradicts to the maximality of Q. Thus the intersection of $\bigcap_{F \in Q} F$ contains only one element, say $g(Q)$. Consequently we have $g(f(p)) = p$ and $f(g(Q)) = Q$. Thus the mappings $f : \mathbb{D} \to X$ and $g : X \to \mathbb{D}$ are bijections inverse to each other. Further, for $A \in P$ we have $g^{-1}(A) = \{Q | \, Q \in \pi, g(Q) \in A\} = \{Q | \, Q \in \pi, Q \in f(A)\} = \{Q | \, Q \in \pi, A \in Q\} = \pi(A)$, which is a subbasic closed set in (X, τ). Then $g : X \to \mathbb{D}$ is continuous. Since X is compact by Theorem 3.1, and $\mathbb{D}$ is Hausdorff (as equippped with the Euclidean topology, and, of course, compact too), the mapping g is a homeomorphism ([2], p. 125, 3.1.13. Theorem). □

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Department of Mathematics, Faculty of Electrical Engineering and Communication, University of Technology, Technická 8, Brno, 616 69, Czech Republic

E-mail address: kovar@feec.vutbr.cz