Solving stochastic linear differential equations in MAPLE

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Stochastic differential equations (SDEs) describe physical systems by taking into account some randomness of the system. A general scalar SDE has the form $\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t)$, where $F : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the drift coefficient and $G : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient. $W(t)$ is the so called Wiener process, a stochastic process representing the noise. (A stochastic process $W(t)$ is called the Wiener process if it has independent increments, $W(0) = 0$ and $W(t) - W(s)$ distributed $N(0, t-s)$, $0 \le s < t$). We can represent the SDE in the integral form

$$X(t) = X(t_0) + \int_{t_0}^{t} F(s, X(s))\ \mathrm{d}s + \int_{t_0}^{t} G(s, X(s))\ \mathrm{d}W(s), \tag{1}$$

where the first integral is an ordinary Riemann integral. Since the sample paths of a Wiener process do not have bounded variation on any time interval, the second integral cannot be a Riemann-Stieljtes integral. K. Itô proposed a way to overcome this difficulty with the definition of a new type of integral, a stochastic integral which is now called the Itô integral (see [4]).

In this paper we consider the scalar linear Itô stochastic differential equation

$$\mathrm{d}X(t) = \big(A_1(t)X(t) + A_2(t)\big)\ \mathrm{d}t + \big(B_1(t)X(t) + B_2(t)\big)\ \mathrm{d}W(t),\ X(0) = X_0 \tag{2}$$

where the coefficients $A_1(t), A_2(t), B_1(t), B_2(t)$ are functions of time or constants. In the case, when $A_2(t) \equiv 0$ and $B_2(t) \equiv 0$, the equation (2) reduces to the homogeneous bilinear Itô SDE. A general solution of a linear stochastic differential equation, like in the case of a deterministic linear differential equation, can be determined explicitly with the help of an integrating factor or a fundamental solution of an associated homogeneous differential equation.

In the paper we present a MAPLE procedure, called Linearsde := proc(a, b), that solves general linear Itô SDEs of the form (2). The input parameters are $a := A_1x + A_2$ and $b := B_1x + B_2$.

Example 1. We want to solve a linear Itô SDE with constant coefficients and with additive noise ($B_1(t) \equiv 0$):

$$\mathrm{d}X(t) = \big(AX(t) + C\big)\ \mathrm{d}t + B\ \mathrm{d}W(t),\ X(0) = 0 \tag{3}$$

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> Linearsde(A*x+C,B);
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$$X[t] = \exp(At)\Big(X[0] + \frac{C(\exp(At) - 1)\exp(-At)}{A} + \int_0^t \frac{B}{\exp(As)}\ \mathrm{d}W\Big)$$

We put $X(0) = 0$ and then simplify this equation. We get, the solution of the equation (3):

$$X(t) = \frac{C}{A}\Big(e^{at} - 1\Big) + B\int_0^t e^{a(t-s)}\ \mathrm{d}W(s)$$

Now we find the solution of (3) using the Itô calculus. We define a function $g(t,x) = e^{-At}x$, and compute its derivative at point $(t, X(t))$ using the Itô formula (see [4]).

$$\mathrm{d}g(t,X(t)) = \mathrm{d}\left(e^{-At}X(t)\right) = e^{-At}(-A)X(t)\,\mathrm{d}t + e^{-At}\,\mathrm{d}X(t) + 0\,(\,\mathrm{d}X(t))^2 =$$

$$-AX(t)e^{-At}\,\mathrm{d}t + e^{-At}\left(\left(AX(t)+C\right)\mathrm{d}t + B\,\mathrm{d}W(t)\right) = Ce^{-At}\,\mathrm{d}t + Be^{-At}\,\mathrm{d}W(t).$$

This in integral form gives us the solution

$$e^{-At}X(t) - X(0) = C\int_0^t e^{-As}\,\mathrm{d}s + B\int_0^t e^{-As}\,\mathrm{d}W(s),$$

and after some trivial computations we get the same solution as in MAPLE

$$X(t) = \frac{C}{A}\left(e^{at}-1\right) + B\int_0^t e^{a(t-s)}\,\mathrm{d}W(s). \tag{4}$$

Example 2. Let us solve the linear Itô SDE:

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + e^{-t}\,\mathrm{d}W(t),\; X(0) = 0. \tag{5}$$

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> Linearsde(-x,exp(-t));
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$$X[t] = \exp(-t)\left(X[0] + \int_0^t 1\,\mathrm{d}W\right)$$

We put $X(0) = 0$ and get the solution of (5) as

$$X(t) = e^{-t}\int_0^t 1\,\mathrm{d}W = e^{-t}\,W(t).$$

Now we use the Itô calculus to find the solution of (5). We define the function $g(t,x) = e^t x$, and compute its derivative at point $(t, X(t))$ using the Itô formula.

$$\mathrm{d}g(t,X(t)) = \mathrm{d}\left(e^tX(t)\right) = e^tX(t)\,\mathrm{d}t + e^t\,\mathrm{d}X(t) = e^tX(t)\,\mathrm{d}t + e^t\left(-X(t)\right)\mathrm{d}t + e^t\,\mathrm{d}W(t) = e^t\,\mathrm{d}W(t).$$

The integral form of this equation is

$$e^tX(t) - X(0) = \int_0^t \mathrm{d}W(s) \quad\Rightarrow\quad X(t) = e^{-t}\,W(t).$$

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