Solving stochastic linear differential equations in MAPLE

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Abstract

This paper deals with linear stochastic differential equations and presents a MAPLE procedure, that solves this types of equations. We present an example, where we solve a special type of linear stochastic differential equation using the MAPLE procedure and then we also solve this equation analytically using the Itô formula.

1 Introduction

Stochastic differential equations (SDEs) describe physical systems by taking into account some randomness of the system. A general scalar SDE has the form $\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t)$, where $F : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the drift coefficient and $G : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient. $W(t)$ is the so called Wiener process, a stochastic process representing the noise. (A stochastic process $W(t)$ is called the Wiener process if it has independent increments, $W(0) = 0$ and $W(t) - W(s)$ distributed $N(0, t - s),\ 0 \le s < t$). We can represent the SDE in the integral form

$$X(t) = X(t_0) + \int_{t_0}^{t} F(s, X(s))\ \mathrm{d}s + \int_{t_0}^{t} G(s, X(s))\ \mathrm{d}W(s), \tag{1}$$

where the first integral is an ordinary Riemann integral. Since the sample paths of a Wiener process do not have bounded variation on any time interval, the second integral cannot be a Riemann-Stieljtes integral. K. Itô proposed a way to overcome this difficulty with the definition of a new type of integral, a stochastic integral which is now called the Itô integral (see [4]).
Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Itô formula, contains an additional term, which, roughly speaking, is due to the fact that the square of the stochastic differential $(\mathrm{d}W(t))^2$ is equal to $\mathrm{d}t$.

The 1-dimensional Itô formula. Let the stochastic process $X(t)$ be a solution of the stochastic differential equation $\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t)$ for some suitable functions F, G (see [4], p.44). Let $g(t, x) : (0, \infty) \times \mathbf{R} \to \mathbf{R}$ be a twice continuously differentiable function. Then

$$Y(t) = g(t, X(t))$$

is a stochastic process, for which

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\ \mathrm{d}t + \frac{\partial g}{\partial x}(t, X(t))\ \mathrm{d}X(t) + \frac{1}{2}\ \frac{\partial^2 g}{\partial x^2}(t, X(t))(\ \mathrm{d}X(t))^2,$$

where $(\ \mathrm{d}X(t))^2 = (\ \mathrm{d}X(t)) \cdot (\ \mathrm{d}X(t))$ is computed according to the rules

$$\mathrm{d}t \cdot\ \mathrm{d}t =\ \mathrm{d}t \cdot\ \mathrm{d}W(t) =\ \mathrm{d}W(t) \cdot\ \mathrm{d}t = 0,\ \ \mathrm{d}W(t) \cdot\ \mathrm{d}W(t) =\ \mathrm{d}t.$$

2 Linear SDEs

The general form of a scalar linear Itô stochastic differential equation is

$$\mathrm{d}X(t) = \big(A_1(t)X(t) + A_2(t)\big)\,\mathrm{d}t + \big(B_1(t)X(t) + B_2(t)\big)\,\mathrm{d}W(t),\ X(0) = X_0 \tag{2}$$

where the coefficients $A_1(t), A_2(t), B_1(t), B_2(t)$ are functions of time or constants. In the case, when $A_2(t) \equiv 0$ and $B_2(t) \equiv 0$, the equation (2) reduces to the homogeneous bilinear Itô SDE. A general solution of a linear stochastic differential equation, like in the case of a deterministic linear differential equation, can be determined explicitly with the help of an integrating factor or a fundamental solution of an associated homogeneous differential equation.

Here we present a MAPLE procedure, that solves general linear Itô SDEs of the form (2). The input parameters are $a := A_1x + A_2$ and $b := B_1x + B_2$.

```
> Linearsde := proc(a,b)
> local temp1,alpha,beta,gamma,delta,fundsoln,fundsoln2,soln,default1,
default2,default3;
> if diff(a,x,x)<>0
> or diff(b,x,x)<>0 then
> ERROR(‘SDE not linear‘)
> else
> alpha:=diff(a,x);
> alpha:=subs(t=s,alpha);
> beta:=diff(b,x);
> beta:=subs(t=s,beta);
> if diff(beta,s)=0 then
> temp1:=beta*W;
> else temp1:=Int(beta,W=0..t);
> fi;
> gamma:=coeff(a,x,0);
> gamma:=subs(t=s,gamma);
> delta:=coeff(b,x,0);
> delta:=subs(t=s,delta);
> fundsoln:=exp(int(alpha-1/2*(beta^2),s=0..t)+temp1);
> fundsoln2:=subs(t=s,fundsoln);
> if beta=0 then
> soln:=fundsoln*(X[0]+int(1/fundsoln2*(gamma-beta*delta),s=0..t)
+Int(1/fundsoln2*delta,W=0..t))
> else
> soln:=fundsoln*(X[0]+Int(1/fundsoln2*(gamma-beta*delta),s=0..t)
+Int(1/fundsoln2*delta,W=0..t))
> fi;
> default1:=Int(0,W=0..t)=0;
> default2:=Int(0,W=0..s)=0;
> default3:=Int(0,s=0..t)=0;
> soln:=X[t]=subs(default1,default2,default3,soln)
> fi
> end:
```

This procedure with many others one can find in [2]. The authors of this book have used MAPLE 5.1, but most of the procedures can also be used with higher version of MAPLE, even with the very last versions. We used MAPLE 11, while preparing this paper.

Example 1. We want to solve a linear Itô SDE with constant coefficients and with additive noise ($B_1(t) \equiv 0$):

$$\mathrm{d}X(t) = \big(AX(t) + C\big)\,\mathrm{d}t + B\,\mathrm{d}W(t),\ X(0) = 0 \tag{3}$$

```
> Linearsde(A*x+C,B);
```

$$X[t] = \exp(At)\Big(X[0] + \frac{C(\exp(At)-1)\exp(-At)}{A} + \int_0^t \frac{B}{\exp(As)}\,\mathrm{d}W\Big)$$

We put $X(0) = 0$ and then simplify this equation. We get, the solution of the equation (3):

$$X(t) = \frac{C}{A}\Big(e^{at} - 1\Big) + B\int_0^t e^{a(t-s)}\,\mathrm{d}W(s)$$

Now we find the solution of (3) using the Itô calculus. We define a function $g(t,x) = e^{-At}x$, and compute its derivative at point $(t, X(t))$ using the Itô formula.

$$\mathrm{d}g(t,X(t)) = \ \mathrm{d}\left(e^{-At}X(t)\right) = e^{-At}(-A)X(t)\,\mathrm{d}t + e^{-At}\,\mathrm{d}X(t) + 0\ (\ \mathrm{d}X(t))^2 =$$

$$-AX(t)e^{-At}\,\mathrm{d}t + e^{-At}\left(\big(AX(t)+C\big)\,\mathrm{d}t + B\,\mathrm{d}W(t)\right) = Ce^{-At}\,\mathrm{d}t + Be^{-At}\,\mathrm{d}W(t).$$

This in integral form gives us the solution

$$e^{-At}X(t) - X(0) = C\int_0^t e^{-As}\,\mathrm{d}s + B\int_0^t e^{-As}\,\mathrm{d}W(s),$$

and after some trivial computations we get the same solution as in MAPLE

$$X(t) = \frac{C}{A}\Big(e^{at} - 1\Big) + B\int_0^t e^{a(t-s)}\,\mathrm{d}W(s). \tag{4}$$

Example 2. Let us solve the linear Itô SDE:

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + e^{-t}\,\mathrm{d}W(t),\ X(0) = 0. \tag{5}$$

```
> Linearsde(-x,exp(-t));
```

$$X[t] = \exp(-t)\left(X[0] + \int_0^t 1\,\mathrm{d}W\right)$$

We put $X(0) = 0$ and get the solution of (5) as

$$X(t) = e^{-t}\int_0^t 1\,\mathrm{d}W = e^{-t}\ W(t).$$

Now we use the Itô calculus to find the solution of (5). We define the function $g(t,x) = e^t x$, and compute its derivative at point $(t, X(t))$ using the Itô formula.

$$\mathrm{d}g(t,X(t)) = \ \mathrm{d}\left(e^tX(t)\right) = e^tX(t)\,\mathrm{d}t + e^t\,\mathrm{d}X(t) = e^tX(t)\,\mathrm{d}t + e^t\big(-X(t)\big)\,\mathrm{d}t + e^t\,\mathrm{d}W(t) = e^t\,\mathrm{d}W(t).$$

The integral form of this equation is

$$e^tX(t) - X(0) = \int_0^t \mathrm{d}W(s) \quad \Rightarrow \quad X(t) = e^{-t}\ W(t).$$

3 The moment equation

If we take the expectation of the integral form of the equation (2) and use the zero expectation property of the Itô integral, we obtain an ordinary differential equation for the expected value $m(t) = E[X(t)]$ of its solution

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = A_1(t)m(t) + A_2(t).$$

We present the procedure moment1, which computes the expectation equation of the solution of (2). The input parameters correspond to the variables of the equation (2).

```
> moment1:=proc(a1,a2)
> diff(m(t),t)=a1*m(t)+a2;
> end:
```

This is an ordinary differential equation, that can be solved in MAPLE.

Example 3. We find the first moment equation and the solution of this equation for the linear Itô SDE (3) with constant coefficients and with additive noise.

```
> moment1(A,C);
```

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = A\ m(t) + C.$$

We find in MAPLE the solution of this ordinary first order differential equation with initial condition $m(0) = E[X(0)] = 0$.

```
> dsolve({%,m(0)=0},m(t));
```

$$m(t) = -\frac{C}{A} + \frac{\exp(At)C}{A}$$

We can compute the expectation directly from the stochastic solution (4) and we get the same solution

$$m(t) = \frac{C}{A}\left(e^{at} - 1\right).$$

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