Stability of a Dynamical Model of Economy with Delay

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We consider a classical dynamical Leontjev model

$$x(t) = Ax(t) + B\dot{x}(t) + c(t), \quad t \geq 0 \tag{1}$$

where x, c are $n\times 1$ real vectors and A, B are real $n\times n$ constant matrices such that $0 \leq a_{ij} < 1$, $\sum_{j=1}^{n} a_{ij} < 1$, $0 \leq b_{ij} < 1$, $\sum_{j=1}^{n} b_{ij} < 1$, $c_i(t) \geq 0$, $i, j = 1, 2, \ldots, n$, $\det A \neq 0$, $\det B \neq 0$. The vector-function c represents investment and, strictly speaking, is a control function. If we expect a solution of (1) $x = \varphi(t)$ and if we suppose its asymptotical stability, then the suggested form of control is

$$c(t) = Cx(t-\tau) + (A-I)\varphi(t) + C\varphi(t-\tau) - B\dot{\varphi}(t)$$

where C is a suitable $n \times n$ constant matrix, $\tau > 0$ is a duration time for transmission of information (information delay) and for construction of the control. Then the system (1) takes the form

$$B\dot{x}(t) + (A-I)x(t) + Cx(t-\tau) = f(t) \tag{2}$$

where $f(t) = B\dot{\varphi}(t) + (A-I)\varphi(t) + C\varphi(t-\tau)$. We denote

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}\,, \quad \varphi(B^T HB) = \frac{\lambda_{\max}(B^T HB)}{\lambda_{\min}(B^T HB)}$$

where $\lambda_{\max}(\,\cdot\,)$ and $\lambda_{\min}(\,\cdot\,)$ are the maximal and the minimal eigenvalue of a positively definite matrix,

$$\|x(t)\|_\tau = \max_{\theta\in[t-\tau,t]} \|x(\theta)\|.$$

Theorem 1 *Assume that a matrix C is given such that the matrix $B^{-1}(I-A-C)$ is asymptotically stable. Let moreover G be a positively definite $n \times n$ matrix and the $n \times n$ matrix H solves the matrix Sylvester equation*

$$(I-A-C)^T HB + B^T H(I-A-C) = -G.$$

Then the system (2) *is asymptotically stable for* $\tau < \tau_0$ *with*

$$\tau_0 = \frac{\lambda_{\min}(G)}{2\|B^T HB\| \cdot [\|B^{-1}(I-A)\| + \|B^{-1}C\|] \sqrt{\varphi(B^T HB)}}.$$

Moreover

$$\|x(t)\| \leq \begin{cases} N\mathrm{e}^{rt}\|x(0)\|_\tau, & 0 \leq t \leq \tau, \\ N\mathrm{e}^{rt}\|x(0)\|_\tau \sqrt{\varphi(B^T HB)}\mathrm{e}^{-\gamma(t-\tau)/2}, & t > \tau \end{cases}$$

where

$$N = 1 + \|B^{-1}C\|\tau, \quad r = \|B^{-1}(I-A)\|,$$

$$\gamma = \frac{(1-\tau/\tau_0)\gamma^*}{\alpha\gamma^* + (1-\tau/\tau_0)}, \quad \gamma^* = \frac{2}{\tau}\ln\left[\frac{\sqrt{c^2+4d}-c}{2d}\right].$$

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