An example of subquasi-order hypergroup

Šárka Hošková
University of Defence Brno, Faculty of Military Technology, Department of Mathematics and Physics, Kounicova 65, Brno
e-mail: sarka.hoskova@unob.cz

Abstract

The aim of this contribution is to give an example of subquasi-order hypergroup. By quasi-order hypergroups (order hypergroups) we mean the hypergroups determined by a binary relation of quasiordering (ordering). This special type of hypergroups were introduced in paper of Jan Chvalina: Commutative hypergroups in the sense of Marty and ordered sets.

In the paper [7] special types of hypergroups, so called quasi-order hypergroups (QOHG) and order hypergroups (OHG), were introduced (cf. also [2, 3, 6, 9]). Recall that a pair \((H, \cdot)\), where \(H\) is a (nonempty) set and \(\cdot : H \times H \rightarrow \mathcal{P} \ast (H) = \mathcal{P}(H) \setminus \{\emptyset\}\) is a binary hyperoperation on the set \(H\) such that \(a \cdot (b \cdot c) = (a \cdot b) \cdot c\) (associativity) and \(a \cdot H = H = H \cdot a\) (the reproduction axiom) is satisfied for all \(a, b, c \in H\), is a hypergroup. Here, for \(A, B \subseteq H, A \neq \emptyset \neq B\) we define as usual \(A \cdot B = \bigcup\{a \cdot b; a \in A, b \in B\}\), (see, e.g. [2]).

Definition 1. A hypergroup \((H, \cdot)\) such that the conditions

(i) \(a \in a^2 = a^3\) for any \(a \in H\),
(ii) \(a \cdot b = a^2 \cup b^2\) for any pair \(a, b \in H\)

are satisfied is called a quasi-order hypergroup. If moreover the unique square root condition

(iii) \(a, b \in H, a^2 = b^2\) implies \(a = b\)

is satisfied, then \((H, \cdot)\) is called an order hypergroup.

It is to be noted that from (i) and (ii) of Definition 1 there follows the extensivity of the hypergroup \((H, \cdot)\), i.e. \(\{x, y\} \subseteq x \cdot y\) for all \(x, y \in H\). For the preceding definition see [7].

In [6] it was shown that the category of all order hypergroups (OHG) forms a full reflective subcategory of category of all quasi-order hypergroups and their inclusion homomorphisms as morphisms (QOHG).

Definition 2. A commutative hypergroup \((H, \cdot)\) such that the conditions

(i) \(a \in a^2 = a^3\),
(ii) \(a \cdot b \subseteq a^2 \cup b^2\),
(iii) \(\{a, b\} \subseteq a \cdot b\)

are satisfied for any pair \(a, b \in H\) will be called a subquasi-order hypergroup.

The category of all subquasi-order hypergroups with inclusive homomorphisms as their morphisms will be denoted \(\text{SQOHG}\). Thus \(\text{QOHG}\) is a full subcategory of \(\text{SQOHG}\).

For \(x \in H\) denote \([x]_\leq\) the upper end determined by \(x\), i.e., \([x]_\leq = \{z \in H; x \leq z\}\).
Example 1. By a modification of some examples contained in paragraph 3, chapt. IV[8] we obtain a large class of suborder hypergroups (or subquasi-order hypergroups). For an arbitrary upper semilattice \((L, \lor, \land)\) or especially a lattice \((L, \lor, \land)\) let us define a binary hyperoperation 
\[
\cdot : L \times L \to \mathcal{P}^*(L) \text{ by } x \cdot y = [x \lor y] \cup \{x, y\},
\]
where “\(\leq\)” is the ordering on \(L\) determined by the join (i.e. supremum) operation “\(\lor\)” or by the usual rule:
\[
x, y \in L, \quad x \leq y \text{ whenever } x \lor y = y \text{ and } x \land y = x.
\]

Then with respect to Lemma 1.13 [5] it is easy to see that \((L, \cdot)\) is a commutative extensive hypergroup, more precisely \((L, \cdot)\) satisfies all conditions from Definition 1.

In particular, if \(S\) is at least a four element set and \((L, \lor, \land) = (\mathcal{P}(S), \cup, \cap)\) then for any pair of singletons \(\{x\}, \{y\} \in \mathcal{P}(S)\) we have \(\{x\} \cdot \{y\} \subset \{x\}^2 \cup \{y\}^2\) and \(\{x\} \cdot \{y\} \neq \{x\}^2 \cup \{y\}^2\). Take e.g. a four element set \(S = \{x, y, u, v\}\). Then
\[
\{x\} \cdot \{u\} = \{\{x\}, \{u\}, \{x, u\}, \{x, u, v\}, \{x, u, v\}\}
\]
\[
\{x\}^2 \cup \{u\}^2 = \{\{x\}, \{u\}, \{x, y\}, \{x, u\}, \{x, v\}, \{y, u\}, \{u, v\}, \{x, u, v\}\}.
\]

So really \(\{x\} \cdot \{y\} \neq \{x\}^2 \cup \{y\}^2\).

Similarly, a subquasi-order hypergroup can be obtained by the sum operation from lattices and quasi-ordered sets which are not ordered sets.

References