An example of subquasi-order hypergroup

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Abstract

The aim of this contribution is to give an example of subquasi-order hypergroup. By quasi-order hypergroups (order hypergroups) we mean the hypergroups determined by a binary relation of quasiordering (ordering). This special type of hypergroups were introduced in paper of Jan Chvalina: *Commutative hypergroups in the sence of Marty and ordered sets.*

In the paper [7] special types of hypergroups, so called *quasi-order hypergroups* ($\mathbb{QOHG}$) and *order hypergroups* ($\mathbb{OHG}$), were introduced (cf. also [2, 3, 6, 9]). Recall that a pair $(H, \cdot)$, where H is a (nonempty) set and $\cdot\colon H \times H \to \mathscr{P}^*(H)\ \ (= \mathscr{P}(H) \smallsetminus \{\emptyset\})$ is a binary hyperoperation on the set H such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity) and $a \cdot H = H = H \cdot a$ (the reproduction axiom) is satisfied for all $a, b, c \in H$, is a *hypergroup*. Here, for $A, B \subseteq H$, $A \neq \emptyset \neq B$ we define as usual $A \cdot B = \bigcup\{a \cdot b; a \in A, b \in B\}$, (see, e.g. [2]).

Definition 1. *A hypergroup $(H, \cdot)$ such that the conditions*

(i) $a \in a^2 = a^3$ *for any* $a \in H$,
(ii) $a \cdot b = a^2 \cup b^2$ *for any pair* $a, b \in H$

are satisfied is called a quasi-order hypergroup. *If moreover the unique square root condition*

(iii) $a, b \in H$, $a^2 = b^2$ *implies* $a = b$

is satisfied, then $(H, \cdot)$ is called an order hypergroup.

It is to be noted that from (i) and (ii) of Definition 1 there follows the extensivity of the hypergroup $(H, \cdot)$, i.e. $\{x, y\} \subset x \cdot y$ for all $x, y \in H$. For the preceding definition see [7].

In [6] it was shown that the category of all *order hypergroups* ($\mathbb{OHG}$) forms a full reflective subcategory of category of all *quasi-order hypergroups* and their inclusion homomorphisms as morphisms ($\mathbb{QOHG}$).

Definition 2. *A commutative hypergroup $(H, \cdot)$ such that the conditions*

(i) $a \in a^2 = a^3$,
(ii) $a \cdot b \subset a^2 \cup b^2$,
(iii) $\{a, b\} \subset a \cdot b$

are satisfied for any pair $a, b \in H$ will be called a subquasi-order hypergroup.

The category of all subquasi-order hypergroups with inclusive homomorphisms as their morphisms will be denoted $\mathbb{SQOHG}$. Thus $\mathbb{QOHG}$ is a full subcategory of $\mathbb{SQOHG}$.

For $x \in H$ denote $[x)_\leq$ the upper end determined by x, i.e., $[x)_\leq = \{z \in H; x \leq z\}$.

Example 1. *By a modification of some examples contained in paragraph 3, chapt. IV[8] we obtain a large class of suborder hypergroups (or subquasi-order hypergroups). For an arbitrary upper semilattice $(L, \vee)$ or especially a lattice $(L, \vee, \wedge)$ let us define a binary hyperoperation*

$$\cdot\colon L \times L \to \mathscr{P}^*(L) \text{ by } x \cdot y = [x \vee y)_{\leq} \cup \{x, y\},$$

where "$\leq$" is the ordering on L determined by the join (i.e. supremum) operation "$\vee$" or by the usual rule:

$$x, y \in L, \quad x \leq y \text{ whenever } x \vee y = y \text{ and } x \wedge y = x.$$

Then with respect to Lemma 1.13 [5] it is easy to see that $(L, \cdot)$ is a commutative extensive hypergroup, more precisely $(L, \cdot)$ satisfies all conditions from Definition 1.

In particular, if S is at least a four element set and $(L, \vee, \wedge) = (\mathscr{P}(S), \cup, \cap)$ then for any pair of singletons $\{x\}, \{y\} \in \mathscr{P}(S)$ we have $\{x\} \cdot \{y\} \subset \{x\}^2 \cup \{y\}^2$ and $\{x\} \cdot \{y\} \neq \{x\}^2 \cup \{y\}^2$. Take e.g. a four element set $S = \{x, y, u, v\}$. Then

$$\begin{aligned}\{x\} \cdot \{u\} &= \big\{\{x\}, \{u\}, \{x, u\}, \{x, y, u\}, \{x, u, v\}, \{x, y, u, v\}\big\}\\ \{x\}^2 \cup \{u\}^2 &= \big\{\{x\}, \{u\}, \{x, y\}, \{x, u\}, \{x, v\}, \{y, u\}, \{u, v\},\\ &\qquad \{x, y, u\}, \{x, y, v\}, \{y, u, v\}, \{x, u, v\}, \{x, y, u, v\}\big\}.\end{aligned}$$

So really $\{x\} \cdot \{y\} \neq \{x\}^2 \cup \{y\}^2$.

Similarly, a subquasi-order hypergroup can be obtained by the sum operation from lattices and quasi-ordered sets which are not ordered sets.

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