Some examples of generated fuzzy implicators

Dana Hliněná and Vladislav Biba
Department of Mathematics, FEEC, Brno University of Technology
Technická 8, 616 00 Brno, Czech Rep.
e-mail: hlinena@feec.vutbr.cz, v_biba@post.sk

Abstract

Conjunctors in MV-logic with truth values range $[0,1]$ are monotone extensions of the classical conjunction. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function, such that $f(1) = 0$, then we can define conjunctor $C : [0,1]^2 \to [0,1]$ by

$$C(x,y) = f^{(-1)}(f(x) + f(y)),$$

where the pseudo-inverse $f^{(-1)}$ is given by $f^{(-1)}(x) = \sup\{t \in [0,1]; f(t) > x\}$, f is called an additive generator of C. A function $I : [0,1]^2 \to [0,1]$ is said to an implicator if and only if $I(1,0) = 0$ and $I(0,0) = I(0,1) = I(1,1) = 1$ and I is non-increasing in its first component and non-decreasing in its second component. The unary operator $n : [0,1] \to [0,1]$ is called negator if for any $a, b \in [0,1]$ holds

$$a \le b \Rightarrow n(b) \le n(a),$$

$$n(0) = 1, n(1) = 0.$$

Starting with the conjunctor C and standard negation $N_s(x) = 1 - x$, we can introduce the implication operator in $[0,1]-$ valued logic as follows: $I_C(x,y) = 1 - C(x, 1-y)$. Another way of extending the classical binary implication operator to the unit interval $[0,1]$ uses the *residuation* R_C with respect to the left-continuous conjunctor C

$$R_C(x,y) = \sup\{z \in [0,1]; C(x,z) \le y\}.$$

There exists several constructions of implicators. We will compare these implicators and some their properties will be given.

1 Introduction

We can recall definitions of the most important connectives in MV-logic.

Definition 1 *An unary operator $n : [0,1] \to [0,1]$ is called negator if for any $a, b \in [0,1]$ it holds*

- *(i)* $a < b \Rightarrow n(b) \le n(a)$,
- *(ii)* $n(0) = 1, n(1) = 0$.

The negator n is called *strong negator* if and only if the mapping n is one to one. Evidently, strong negator is continuous and its inverse n^{-1} is strong negator too. The negator n is called *involutive negator* if and only if for all $a \in [0,1]$, $n(n(a)) = a$. It can be easily proved that involutive negator is strong and $n^{-1} = n$.

Example 1
- *(1)* $n(a) = 1 - a$ *involutive negator,*

- *(2)* $n(a) = 1 - a^2$ *strong, non-involutive negator,*
- *(3)* $n(a) = \sqrt{1 - a^2}$ *involutive negator,*
- *(4)* $n(0) = 1, n(a) = 0$ *if* $a > 0$ *non-strong negator.*

Definition 2 *A non-decreasing mapping* $C : [0,1]^2 \rightarrow [0,1]$ *is called conjunctor if for any* $a, b \in [0,1]$ *it holds*

- *(i)* $C(a,b) = 0$ *whenever* $a = 0,$ *or* $b = 0$
- *(ii)* $C(1,1) = 1.$

Commonly used conjunctors in MV-logic are the triangular norms.

Definition 3 *A triangular norm (t-norm for short) is a binary operation on the unit interval* $[0,1]$*, i.e., a function* $T : [0,1]^2 \rightarrow [0,1]$ *such that for all* $x, y, z \in [0,1]$ *the following four axioms are satisfied:*
(T1) Commutativity $T(x,y) = T(y,x),$
(T2) Associativity $T(x, T(y,z)) = T(T(x,y), z),$
(T3) Monotonicity $T(x,y) \leq T(x,z)$ *whenever* $y \leq z,$
(T4) Boundary Condition $T(x,1) = x.$

Example 2 *The following are the four basic t-norms:*

- *Minimum* T_M *given by*
$$T_M(x,y) = \min(x,y),$$
- *Product* T_P *given by*
$$T_P(x,y) = x \cdot y,$$
- *Lukasiewicz t-norm* T_L *given by*
$$T_L(x,y) = \max(0, x + y - 1),$$
- *Drastic product* T_D *given by*
$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1 *Note, that the dual operator to the conjunctor* C *defined by a non-decreasing mapping* $D : [0,1]^2 \rightarrow [0,1]$*, such that* $D(a,b) = 1$ *whenever* $a = 1$ *or* $b = 1$ *and* $D(0,0) = 0$ *is called the disjunctor* D*. Commonly used disjunctors in MV-logic are the triangular conorms. Triangular conorms (also called* $S-$*norms) are dual to* $t-$*norms under the order reversing operation which assigns* $1 - x$ *to* x *on* $[0,1]$*.*

Definition 4 *A function* $I : [0,1]^2 \rightarrow [0,1]$ *is said to be an implicator if* $I(1,0) = 0,$ $I(0,0) = I(0,1) = I(1,1) = 1,$ I *is non-increasing in its first component and non-decreasing in its second component.*

Starting with the conjunctor C and the negation n, we can introduce the implication operator in $[0,1]$-valued logic as follows:

$$I_C(x,y) = n(C(x,n(y))).$$

Another way of extending the classical binary implication operator to the unit interval $[0,1]$ uses the *residuation* R_C with respect to a left-continuous conjunctor C

$$R_C(x,y) = \sup\{z \in [0,1]; C(x,z) \leq y\}.$$

When starting with the implicator I we can define a conjunctor C_I as follows

$$C_I(x,y) = \inf\{z \in [0,1]; I(x,z) \geq y\}.$$

Remark 2 *Note that in the classical logic it is $C = C_{I_C}$ but in MV-logic C is not equal to C_{I_C} in general.*

Our constructions of implicators will make use of extending the classical inverse of function. One way of extending is described in next definitions.

Definition 5 *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function. The function $\varphi^{(-1)}$ which is defined by*

$$\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\},$$

is called the pseudo-inverse of the function φ, with the convention $\sup\emptyset = 0$.

Definition 6 *Let $f : [0,1] \to [0,1]$ be a non-increasing function. The function $f^{(-1)}$ which is defined by*

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\},$$

is called the pseudo-inverse of the function f, with the convention $\sup\emptyset = 0$.

2 The construction of implicators based on generators

There exist several constructions of implicators via generalized conjunctors and disjunctors. Detailed describtion of these constructions are in [4]. Main contributions of this paper are new properties of these generalized implicators which are described in next propositions.

Let f be a strictly decreasing function such that $f(1) = 0$ and g be a strictly increasing function such that $g(0) = 0$. Then we can define the implicators I_f, I^g as follows:

$$I_f(x,y) = f^{(-1)}(f(y^+) - f(x)),$$

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)),$$

where $f(y^+) = \lim\limits_{x \to y^+} f(x)$. Now, we recall definitions of some important properties of implicators which we will investigate.

Definition 7 *An implicator I is called border implicator if for all $b \in [0,1]$ it holds*

$$I(1,b) = b.$$

Definition 8 *An implicator I is said to satisfy the exchange principle, if*

$$I(x, I(y, z)) = I(y, I(x, z) \text{ for all } x, y, z \in [0, 1].$$

Definition 9 *A border implicator is called contrapositive implicator with respect to a given negator n if for all $a, b \in [0, 1]$ it holds*

$$I(a, b) = I(n(b), n(a)).$$

The main contributions of our paper are, infact corrollaries of the following technical result, which has not been published to our knowledge yet.

Proposition 1 *Let c be a positive real number, then for pseudo-inverse of positive multiple of any left-continuous function f we get*

$$(c \cdot f(x))^{(-1)} = f^{(-1)}\left(\frac{x}{c}\right).$$

Proof. Let f be a non-decreasing function, then

$$f^{(-1)}(x) = \sup\{z \in \ [0, 1]; f(z) < \ x\}$$

and then

$$(c \cdot f)^{(-1)}(x) = \sup\{z \in \ [0, 1]; c \cdot f(z) < \ x\} = \sup\left\{z \in \ [0, 1]; f(z) < \ \frac{x}{c}\right\} = f^{(-1)}\left(\frac{x}{c}\right).$$

Now, the proof for the case of non-increasing function is analogous. ∎

First, we will investigate the properties of I_f implicators:

Proposition 2 *Let $f : [0, 1] \to [0, \infty]$ be a left-continuous, strictly decreasing function such that $f(1) = 0$. Then I_f is border implicator and moreover $I_f = R_C$, where C is the conjunctor generated by additive generator f.*

It is well known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also true for the f generators of I_f implicators. The next theorem is a corrollary of Proposition 1.

Theorem 1 *The f generator of an I_f implicator is uniquely determined up to a positive multiplicative constant.*

Second, we turn our attention to I^g implicators and their properties.

Proposition 3 *Let $g : [0, 1] \to [0, \infty]$ be a left-continuous, strictly increasing function such that $g(0) = 0$. Then I^g is border and contrapositive implicator and moreover $I^g = R_{C^*}$, where C^* is the conjunctor generated by additive generator f^*, $f^*(x) = g(1 - x)$.*

Remark 3 *Note, that if $f(x) = g(1 - x)$, then implicators I_f and I^g are identical.*

Proposition 4 *Let g be a left-continuous, strictly increasing function such that $g(0) = 0$. Then implicator I^g satisfies the exchange principle.*

Theorem 2 *The g generator of an I^g implicator is uniquely determined up to a positive multiplicative constant.*

Generalization of I^g implicators is given in next proposition.

Proposition 5 *Let n be a negator, g be a left-continuous, strictly increasing function such that* $g(0) = 0$*. Then the function* $I_n^g : [0,1]^2 \to [0,1]$ *which is defined by*

$$I_n^g(x, y) = g^{(-1)}(g(n(x)) + g(y)),$$

is border implicator.

Proposition 6 *Let n be an involutive negator, g be a left-continuous, strictly increasing function such that* $g(0) = 0$*. Then implicator* I_n^g *is contrapositive implicator with respect to the negator* n*.*

Proposition 7 *Let g be a left-continuous, strictly increasing function such that* $g(0) = 0$*. Then implicator* I_n^g *holds the exchange principle.*

Theorem 3 *The g generator of an* I_n^g *implicator is uniquely determined up to a positive multiplicative constant.*

Acknowledgement.
Supported by Project 1ET100300517 of the Program "Information Society" and by Project MSM0021630529 of the Ministry of Education.

References

[1] B. De Baets, R. Mesiar. Residual implicators of continuous t-norms *Proc. EUFIT '96, Aachen*, 27-31, 1996

[2] S. Gottwald. Fuzzy Sets and Fuzzy Logic *Vieweg, Braunschweig*, 1993

[3] R. Mesiar. Generated conjunctors and related operators in MV-logic as a basis for AI applications *ECAI'98 Workshop 17, Brighton*, 1–5, 1998

[4] D. Smutná. On many valued conjunctions and implications *Journal of Electrical Engineering* **10/s** vol. **50**, 8–10, 1999