On non-periodic homogenization

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1 Introduction

The model homogenization problem deals with behavior as $\varepsilon \to 0$ of solutions to a sequence of elliptic equations

$$-\mathrm{div}\,(a^\varepsilon(x)\,\nabla u_\varepsilon) = f \tag{1}$$

in a domain Ω in $\mathbb{R}^N$ completed by a suitable boundary conditions e.g. $u^\varepsilon = 0$ on $\partial\Omega$. The matrix of coefficients $a^\varepsilon \equiv (a^\varepsilon_{ij}(x))_{i,j}$ are indexed by a sequence $\{\varepsilon\}$ of small positive parameters ε_k converging to zero, the subscript k in ε_k being omitted.

In the classical periodic homogenization with basic cell $Y = \langle 0,1)^N$ the coefficients a^ε are defined by a Y-periodic function $a(y)$ — i.e. $a(y+k) = a(y)$ for each $k \in \mathbb{Z}^N$ — by setting

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right).$$

The coefficients a^ε_{ij} form a sequence of periodic functions with diminishing period ε.

The are several deterministic generalization, e.g. the oscillations need not be uniform, i.e. $a^\varepsilon(x) = a\,(x, x/\varepsilon)$, where $a(x,y)$ is periodic in y; the coefficients $a(x,y)$ may be almost-periodic functions in y; the coefficients may oscillates with two different periodic scales (reiterated homogenization), e.g. $a^\varepsilon(x) = a\left(x, x/\varepsilon, x/\varepsilon^2\right)$. And we do not mention the stochastic coefficients.

In the contribution we shall try to outline a new approach introduced by G. Nguetseng in [1] which aims to cover all deterministic cases.

2 Homogenization structure and auxiliary results

The basic concept is called Homogenization structure. Let us denote by Π the space of all bounded continuous functions on $\mathbb{R}^N$ which are ponderable, i.e. functions u having the mean value $M(u)$ defined as a $L^\infty(\mathbb{R}^N)$-weak* limit of $u^\varepsilon(x) = u(x/\varepsilon)$ as $\varepsilon \to 0$. The space Π with the supremum norm is a Banach space. In addition, it is a commutative Banach algebra with unit and multiplication defined by $(u \cdot v)(x) = u(x) \cdot v(x)$.

A separable multiplicative subgroup Γ of Π is called *H-structure.* It generates a Banach algebra $A = A_\Sigma$ containing constant functions. This subalgebra in Π will be called *H-algebra.*

Spectrum $\Delta(A)$ of the algebra A, see [2], is a set of all nonzero multiplicative linear forms on A. It is a subset of the dual A^*. It can be identified also with all proper maximal ideals of the algebra A. In case when A is the algebra of all continuous Y-periodic functions, the spectrum $\Delta(A)$ can be identified with the period Y.

The Gelfand representation $\mathcal{G} : A \to \mathcal{C}(\Delta A)$ assigns to a function $a \in A$ a continuous function $\widehat{a}$ on $\Delta(A)$ by relation $\widehat{a}(f) = f(a)$ for all $f \in \Delta(A)$. With the weak product topology the spectrum $\Delta(A)$ is a compact metric space.

The mean value mapping M generates unique Radon measure β on $\Delta(A)$ such that for $u \in A$

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)(s)\mathrm{d}\beta(s).$$

The closure of the Banach algebra $A = A_\Sigma$ in the norm $\sup_{0<\varepsilon\leq 1}(\int_{|x|<1} |u(\frac{x}{\varepsilon})|^p)^{1/p}$ is a Banach space denoted by $\mathcal{X}^p_\Sigma$ and the Gelfand mapping can be extended to $\mathcal{G} : \mathcal{X}^p_\Sigma \to L^p(\Delta(A))$ and introduce Lebesgue spaces on $\Delta(A)$. Similarly the subspace of smooth functions in A enables to differentiate continuous functions of $\mathcal{C}(\Delta(A))$ and define a Sobolev-type space $H^1(\Delta(A))$.

The main tool for getting the result is a generalization of 2-scale convergence:

Definition *Sequence $\{u_\varepsilon\}_\varepsilon$ in $L^2(\Omega)$ is said to weakly Σ-converge to an $u_0 \in L^2(\Omega, \Delta(A))$ if*

$$\int_\Omega u_\varepsilon(x) v^\varepsilon(x) \mathrm{d}x \to \iint_{\Omega\times\Delta(A)} u_0(x,s)\widehat{v}(x,s)\mathrm{d}x\, \mathrm{d}\beta(s)$$

for each $v \in L^2(\Omega; A)$, where $v^\varepsilon(x) = v(x, x/\varepsilon)$ and $\widehat{v} = \mathcal{G} \circ v$.

The convergence brings a compactness: each sequence u_ε bounded in $L^2(\Omega)$ contains a subsequence $u_{\varepsilon'}$ weakly Σ-converging to an $u_0 \in L^2(\Omega, \Delta(A))$. A stronger version is called *strong Σ-convergence*.

Finally an H-structure Σ is called to be *proper* if it satisfies some density, regularity and reflexivity conditions. The H-structures of periodic and almost periodic functions are proper.

3 Homogenization problem

For each $\varepsilon > 0$ the solution $u_\varepsilon \in H^1_0(\Omega)$ is supposed to satisfy the equation (1) with $f \in H^{-1}(\Omega)$ in a bounded domain Ω of $\mathbb{R}^N$. The coefficients are given by $a^\varepsilon_{ij}(x) = a_{ij}(x, x/\varepsilon)$, where a_{ij} are symmetric and satisfies ellipticity condition.

In this setting let Σ be a proper H-structure on $\mathbb{R}^N$ and assume $a_{ij}(x, \cdot) \in \mathcal{X}^2_\Sigma$ for each $x \in \overline{\Omega}$. We put $\mathbb{F}^1_0 = H^1_0(\Omega) \times L^2(\Omega; H^1_\sharp(\Delta(A))$, where the H-algebra A is generated by Σ. Using $\widehat{a}_{ij} = \mathcal{G}(a_{ij}) \in C(\overline{\Omega}, L^\infty(\Delta))$ we define a bilinear elliptic form $\widehat{a}_\Omega$ on $\mathbb{F}^1_0 \times \mathbb{F}^1_0$ for the cell problem. Then using strong Σ convergence of the coefficients and weak Σ convergence the the solutions the proof can follow the idea of the proof based on 2-scale convergence. In this case the homogenized coefficients of the homogenized equation $-\mathrm{div}\,(q(x)\,\nabla u^*) = f$ are given by

$$q_{ij}(x) = \int_{\Delta(x)} \left[\widehat{a}_{ij}(x,s) - \sum_k \widehat{a}_{ik}(x,s)\partial_k \chi^j(x,s)\right] \mathrm{d}\beta(s)$$

where χ^j are solutions to a “cell” problem on $\Delta(A)$.

4 Conclusions

The outlined approach introduced by Nguetseng in [1] uses deep results of functional analysis of Banach algebras. It seems to solve the problem of finding proper space for test functions in the classical two-scale convergence. Moreover, it seems to cover deterministic non-periodic homogenization problems. It can be generalized to reiterated homogenization.

Acknowledgement. This research was supported by the Grant Agency of the Czech Republic, grant No. 201/08/0874.

References

[1] G. Nguetseng. Homogenization structures ans applications *Z. Anal. Anwendungen*, part I: **22**: 73-107, 2003 and part II: **23**: 482-508, 2004.

[2] R. Larsen. *Banach Algebras.* Marcel Dekker, New York, 1973.