Singular initial problem for nonlinear integrodifferential equations of Fredholm-Volterra type

Olga Filippova
Faculty of Electrical Engineering and Communication, VUT Brno, Technická 8, Czech Republic
e-mail: xfilip03@stud.feec.vutbr.cz

Zdeněk Šmarda
Faculty of Electrical Engineering and Communication, VUT Brno, Technická 8, Czech Republic
e-mail: smarda@feec.vutbr.cz

In the past few years, many articles are devoted to the investigation of various singular initial problems for ordinary and integrodifferential equations under different conditions on the nonlinearity and the kernel (see [1-7].) The fundamental tools used in the existence proofs of all above mentioned works are essentially Schauder's theorem in [6], Schauder-Tychonoff's theorem in [1], Banach fixed point theorem in [5], monotone iterative combined with upper and lower solutions in [4,] and Wazewski's topological principle in [2,3,6].
In this paper we give sufficient conditions for existence and uniqueness of solutions of the following singular initial value problem

$$y'(t) = \mathcal{F}\left(t, y(t), \int_{0^+}^{t} K_1(t,s,y(s))ds, \int_{0^+}^{1} K_2(t,s,y(s)ds, \mu\right), \quad y(0^+,\mu) = 0, \tag{1}$$

and, moreover, we shall also investigate a problem of continuous dependence of solutions on a parameter.

Suppose

(I) $\mathcal{F} : \Omega \to R^n$, $\mathcal{F} \in C^0(\Omega)$,
$\Omega = \{(t, u_1, u_2, u_3, \mu) \in J \times (R^n)^3 \times R : |u_1| \le \phi(t),\ |u_2| \le \psi(t),\ |u_3| \le \psi(t)\},\ \ J = (0,1]$,
$0 < \phi(t) \in C^0(J),\ \phi(0^+) = 0,\ 0 < \psi(t) \in C^0(J),\ \ |\cdot|$ denotes the usual norm in R^n,
$|\mathcal{F}(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \mu) - \mathcal{F}(t, \overline{\overline{u}}_1, \overline{\overline{u}}_2, \overline{\overline{u}}_3, \mu)| \le \sum_{i=1}^{3} M_i |\overline{u}_i - \overline{\overline{u}}_i|$ for all $(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \mu)$, $(t, \overline{\overline{u}}_1, \overline{\overline{u}}_2 \overline{\overline{u}}_3, \mu)$ $\in \Omega,\ M_i \ge 0,\ i = 1,2,3.$

(II) $K_j : \Omega^1 \to R^n$, $K_j \in C^0(\Omega_1)$,
$\Omega_1 = \{(t,s,v) \in J \times J \times R^n : |v| \le \phi(t)\ \},\ |K_1(t,s,\overline{v}) - K_1(t,s,\overline{\overline{v}})| \le N_1 |\overline{v} - \overline{\overline{v}}\ |$,
$|K_2(t,s,\overline{v}) - K_2(t,s,\overline{\overline{v}})| \le N_2 e^{\lambda(t-s)} |\overline{v} - \overline{\overline{v}}|$ for all $(t,s,\overline{v})$, $(t,s,\overline{\overline{v}}) \in \Omega_1$, $N_j \ge 0,\ j = 1,2.$
$\lambda > 0$ is a sufficiently large constant such that

$$\left(\frac{M_1 + M_3 N_2}{\lambda} + \frac{M_2 N_1}{\lambda^2}\right) < 1.$$

Theorem 1 *Let the functions $\mathcal{F}(t, u_1, u_2, u_3, \mu)$, $K_j(t,s,v)$, $j = 1,2$ satisfy conditions (I), (II) and, moreover*

$$|\mathcal{F}| \le \sum_{i=1}^{3} g_i(t)|u_i|, \quad 0 < g_i(t) \in C^0(J), \quad \int_{0^+}^{t} g_1(s)\phi(s)ds \le \alpha\phi(t),$$

$$\int_{0^+}^{t} (g_2(s) + g_3(s))\psi(s)ds \leq \beta\phi(t), \quad \alpha + \beta \leq 1,$$

then the problem (1) has a unique solution $y(t, \mu)$ *for each* $\mu \in R, \ t \in J$.

Theorem 2 *Let the assumptions of Theorem 2.1 be satisfied and let there exist a constant* $L > 0$ *and the integrable function* $\gamma : J_0 \to J_0$, *such that*

$$|\mathcal{F}(t, u_1, u_2, u_3, \mu_2) - \mathcal{F}(t, u_1, u_2, u_3, \mu_1)| \leq \gamma(t)|\mu_2 - \mu_1|,$$

where $(t, u_1, u_2, u_3, \mu_1), \ (t, u_1, u_2, u_3, \mu_2) \in \Omega$ *and*

$$\max_{t \in J_0} \left\{ e^{-\lambda t} \int_{0^+}^{t} \gamma(s)ds \right\} \leq L,$$

then the solution $y(t, \mu)$ *of (1) is continuous with respect to the variables* $(t, \mu) \in J \times R$.

Acknowledgement

This research has been supported by the Czech Ministry of Education in the frames of MSM0021 60503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies and MSM0021630529 Research Intention Inteligent Systems in Automatization.

References

[1] Benchohra, M., *On initial value problem for an integrodifferential equation,* Ann. Stiint. Univ. AL.I.CUZA IASI ,XV, (1999), 297-304.

[2] Diblík, J., Ružičková, M., *Existence of positive solutions of a singular initial problem for a nonlinear system of differential equations,* Rocky Mountain Journal of Mathematics **34**, (2004), 923-944.

[3] Diblík, J., Nowak, Ch., *A nonuniqueness criterion for a singular system of two ordinary differential equations,* Nonlinear Analysis, **64** (2006), 637-656.

[4] O'Regan, D., *Nonresonant nonlinear singular problems in the limit circle case,* J.Math. Anal. Appl. **197**, (1996), 708-725.

[5] Šmarda, Z., *On solutions of an implicit singular system of integrodifferential equations depending on a parameter*, Demonstratio Mathematica, Vol.XXXI, No 1, (1998), 125-130.

[6] Šmarda, Z., *On an initial value problem for singular integro- differential equations*, Demonstratio Mathematica, Vol. XXXV, No 4, (2002), 803-811.

[7] Šmarda, Z., *Existence and uniqueness of solutions of nonlinear integrodifferential equations,* Journal of Applied Mathematics, Statistics and Informatics, **2**, (2005), 73-79.