Applications of integral inequalities

Břetislav Fajmon
Faculty of Electrical Engineering and Communication, VUT Brno, Technická 8, Czech Republic
e-mail: fajmon@feec.vutbr.cz

Zdeněk Šmarda
Faculty of Electrical Engineering and Communication, VUT Brno, Technická 8, Czech Republic
e-mail: smarda@feec.vutbr.cz

In this paper we present applications of the inequalities established by Pachpatte[1] which we can formulate as the following theorem:

Theorem 1 *Let $u,\ f,\ g,\ h$ be nonnegative continuous functions defined on R^+ and c be a nonnegative constant.*

(I) If

$$u^2(t) \le c^2 + 2\int_0^t \left[f(s)u(s)\left(u(s) + \int_0^s g(\tau)u(\tau)d\tau \right) + h(s)u(s) \right] ds,$$

for $t \in R^+$, then

$$u(t) \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s [f(\tau)+g(\tau)]d\tau \right) ds \right]$$

where

$$p(t) = c + \int_0^t h(s)ds \tag{1}$$

for $t \in R^+$.

(II) If

$$u^2(t) \le c^2 + 2\int_0^t \left[f(s)u(s)\left(\int_0^s g(\tau)u(\tau)d\tau \right) + h(s)u(s) \right] ds,$$

for $t \in R^+$, then

$$u(t) \le p(t)\exp\left(\int_0^t f(s)\left(\int_0^s g(\tau)d\tau \right) ds \right),$$

for $t \in R^+$, where $p(t)$ is defined by (1)

Pachpatte [2] applied these inequalities to obtain bounds on solutions of some classes ordinary differential equations. We extend his results to a nonlinear integrodifferential equation of the form

$$x'(t) - F\left(t, x(t), \int_0^t k(t,s,x(s))ds \right) = h(t), \quad x(0) = x_0, \tag{2}$$

where $h : R^+ \to R,\ k : R^+ \times R^+ \times R \to R,\ F : R^+ \times R^2 \to R$ are continuous functions . Here we assume that the solution $x(t)$ of (2) exists on R^+. Multiplying both sides of equation (2) by $x(t)$, substituting $t = s$ and then integrating from 0 to t we have

$$x^2(t) = x_0^2 + 2\int_0^t \left[x(s)F\left(s, x(s), \int_0^s k(s,,\tau,x(\tau))d\tau \right) + h(s)x(s) \right] ds. \tag{3}$$

Suppose that

$$|k(t, s, x(s))| \leq f(t)g(s)|x(s)|, \tag{4}$$

$$|F(t, x(t), v)| \leq f(t)|x(t)| + |v|, \tag{5}$$

where f and g are real-valued nonnegative continuous functions defined on R^+. From (3)-(5) we get

$$|x(t)|^2 \leq |x_0|^2 + 2\int_0^t \left[f(s)|x(s)| \left(|x(s)| + \int_0^s g(\tau)|x(\tau)|d\tau \right) + |h(s)||x(s)| \right] ds. \tag{6}$$

Theorem 2 *Let assumptions* (4), (5) *hold then*

$$|x(t)| \leq p_1(t) \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]d\tau \right) ds \right], \tag{7}$$

where

$$|p_1(t) = |x_0| + \int_0^t |h(s)|ds,$$

for $t \in R^+$ *.*

The inequality (7) gives the bound on the solution $x(t)$ of equation (2) in terms of the known functions.

Acknowledgement

This research has been supported by the Czech Ministry of Education in the frames of MSM0021 60503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies and MSM0021630529 Research Intention Inteligent Systems in Automatization.

References

[1] Pachpatte, B.G.: *On some new inequalities related to certain inequalities in the theory of differential equations,* J.Math. Anal. Appl., vol.189, 1995, 128-144.

[2] Pachpatte, B.G.: *Inequalities for differential and integral equations,* Mathematics in Science and Engineering, vol.197, ACADEMIC PRESS INC , 2006, pp.609.