On a Discrete System of Verhulst's Type

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In this contribution we investigate a special system of two discrete equations:

$$\begin{aligned} \Delta u_1(k) &= u_2(k)\left(\beta_1(k) - \gamma_1(k)u_1(k)\right), \\ \Delta u_2(k) &= u_1(k)\left(\beta_2(k) - \gamma_2(k)u_2(k)\right) \end{aligned} \tag{1}$$

where $k \in \mathbb{Z}_a^\infty := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $\gamma_i, \beta_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}^+ := (0, \infty)$, $i = 1, 2$. This system is similar to the scalar equation

$$\Delta u(k) = u(k)\left(\beta(k) - \gamma(k)u(k)\right)$$

which is called (with regard to terminology used in [1]) the Verhulst's equation. Main attention is paid to the asymptotic behavior of solutions of the system. The presented result for system (1) can be obtained from a general result published in [2] which describes sufficient conditions that give the guarantee that all solutions of system (2) (see below) starting at a point in a given domain stay in this domain.

In [2], the general form of the studied system is

$$\Delta u(k) = F(k, u(k)) \tag{2}$$

with $u = (u_1, \dots, u_n)$ and $F = (F_1, \dots, F_n) : \mathbb{Z}_a^\infty \times \mathbb{R}^n \to \mathbb{R}^n$.

We present here the result of paper [2] in a slightly modified form to avoid the introduction of too many new notions.

Theorem 1 *Let $b_i, c_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $i = 1, \dots, n$, be functions such that $b_i(k) < c_i(k)$ for each $k \in \mathbb{Z}_a^\infty$ and suppose that for all the points $M = (k, u_1, \dots, u_n)$, $k \in \mathbb{Z}_a^\infty$, $b_i(k) \le u_i \le c_i(k)$, $i = 1, \dots, n$, the following conditions hold:*
If $u_i = b_i(k)$ for some $i \in \{1, \dots, n\}$, then

$$b_i(k+1) < b_i(k) + F_i(M) < c_i(k+1). \tag{3}$$

If $u_i = c_i(k)$ for some $i \in \{1, \dots, n\}$, then

$$b_i(k+1) < c_i(k) + F_i(M) < c_i(k+1). \tag{4}$$

Let, moreover, the functions

$$G_i(w) := w + F_i(k, u_1, \dots, u_{i-1}, w, u_{i+1}, \dots, u_n), \quad i = 1, \dots, n, \tag{5}$$

be monotone on $[b_i(k), c_i(k)]$ *for every fixed* $k \in \mathbb{Z}_a^\infty$ *and every fixed* $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$, $u_j \in [b_j(k), c_j(k)]$, $j = 1, \dots, i-1, i+1, \dots, n$.
Then for any initial condition $u(a) = u^a = (u_1^a, \dots, u_n^a)$ *with* $b_i(a) < u_i^a < c_i(a)$, $i = 1, \dots, n$, *the corresponding solution* $u = u^*(k) = (u_1^*(k), \dots, u_n^*(k))$ *of system* (2) *satisfies the inequalities*

$$b_i(k) < u_i^*(k) < c_i(k) \tag{6}$$

for every $k \in \mathbb{Z}_a^\infty$.

Theorem 1 can be applied to system (1). Let us denote

$$\omega_i(k) = \frac{\beta_i(k)}{\gamma_i(k)}, \quad i = 1, 2,$$

and

$$f_1(k) = -\frac{\Delta\omega_1(k)}{\omega_2(k)\gamma_1(k)}, \quad f_2(k) = -\frac{\Delta\omega_2(k)}{\omega_1(k)\gamma_2(k)}.$$

Theorem 2 *Suppose that for every* $k \in \mathbb{Z}_a^\infty$ *the following assumptions hold:*

1) $\Delta\omega_i(k) < 0$ *for* $i = 1, 2$.

2) $\Delta\omega_i(k) + f_i(k+1) > 0$ *for* $i = 1, 2$.

3) $f_1(k) + f_2(k)(\Delta\omega_1(k))/\omega_2(k) > 0$ *and* $f_2(k) + f_1(k)(\Delta\omega_2(k))/\omega_1(k) > 0$.

4) $\Delta f_i(k) > 0$ *for* $i = 1, 2$.

Then for any initial condition $u(a) = (u_1^a, u_2^a)$ *with*

$$\omega_i(a) < u_i^a(k) < \omega_i(a) + f_i(a), \quad i = 1, 2,$$

the corresponding solution $u = u^*(k) = (u_1^*(k), u_2^*(k))$ *of system* (1) *satisfies the inequalities*

$$\omega_i(k) < u_i^*(k) < \omega_i(k) + f_i(k) \tag{7}$$

for $i = 1, 2$ *and* $k \in \mathbb{Z}_a^\infty$.

The proof of this theorem is done by showing that all the assumptions of Theorem 1 are satisfied.

Acknowledgment

The first author was supported by the Council of Czech Government MSM 00216 305 03 and MSM 00216 305 19, and by the Grant 201/07/0145 of Czech Grant Agency (Prague).

References

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