On a Discrete System of Verhulst's Type

Josef Diblík
Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics and Descriptive Geometry
e-mail: diblik.j@fce.vutbr.cz

Irena Hlavičková
Brno University of Technology, Faculty of Electrical Engineering and Communication,
Department of Mathematics
e-mail: hlavicka@feec.vutbr.cz

Abstract

In this contribution a special system of two discrete equations is investigated. The studied equations are similar to the well known Verhulst's equation. Main attention is paid to the asymptotic behavior of solutions of the system. Previous results of the authors are used to show that there exists a family of solutions whose graphs stay in a prescribed domain.

1 Introduction

In recent works of the authors ([3]–[6], [9]) the asymptotic behavior of solutions of systems of difference equations is studied. The basic form of such system is

$$\Delta u(k) = F(k, u(k)) \tag{1}$$

with $k \in \mathbb{Z}_a^\infty := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $u = (u_1, \dots, u_n)$, $\Delta u(k) = u(k+1) - u(k)$ and $F = (F_1, \dots, F_n) : \mathbb{Z}_a^\infty \times \mathbb{R}^n \to \mathbb{R}^n$.

If we prescribe an initial condition

$$u(a) = u^a = (u_1^a, \dots, u_n^a) \in \mathbb{R}^n, \tag{2}$$

then the initial problem (1), (2) has a unique solution defined on $\mathbb{Z}_a^\infty$.

In paper [5] one can find sufficient conditions with respect to the right-hand side of system (1) that give the guarantee that all solutions of system (1) starting at a point in a given domain stay in this domain. Here this result is applied to the investigation of the asymptotic behavior of solutions of the system

$$\begin{aligned} \Delta u_1(k) &= u_2(k)\left(\beta_1(k) - \gamma_1(k) u_1(k)\right), \\ \Delta u_2(k) &= u_1(k)\left(\beta_2(k) - \gamma_2(k) u_2(k)\right) \end{aligned} \tag{3}$$

where $\gamma_i, \beta_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}^+ := (0, \infty)$, $i = 1, 2$. This system is similar to the scalar equation

$$\Delta u(k) = u(k)\left(\beta(k) - \gamma(k) u(k)\right)$$

which is called (with regard to terminology used in [1]) the Verhulst's equation.

We present here the result of paper [5] in a slightly modified form to avoid the introduction of too many new notions.

Theorem 1 *Let $b_i, c_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $i = 1, \dots, n$, be functions such that $b_i(k) < c_i(k)$ for each $k \in \mathbb{Z}_a^\infty$ and suppose that for all the points $M = (k, u_1, \dots, u_n)$, $k \in \mathbb{Z}_a^\infty$, $b_i(k) \le u_i \le c_i(k)$, $i = 1, \dots, n$, the following conditions hold:*
If $u_i = b_i(k)$ for some $i \in \{1, \dots, n\}$, then

$$b_i(k+1) < b_i(k) + F_i(M) < c_i(k+1). \tag{4}$$

If $u_i = c_i(k)$ for some $i \in \{1, \dots, n\}$, then

$$b_i(k+1) < c_i(k) + F_i(M) < c_i(k+1). \tag{5}$$

Let, moreover, the functions

$$G_i(w) := w + F_i(k, u_1, \dots, u_{i-1}, w, u_{i+1}, \dots, u_n), \quad i = 1, \dots, n, \tag{6}$$

be monotone on $[b_i(k), c_i(k)]$ for every fixed $k \in \mathbb{Z}_a^\infty$ and every fixed $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$, $u_j \in [b_j(k), c_j(k)]$, $j = 1, \dots, i-1, i+1, \dots, n$.
Then for any initial condition (2) with $b_i(a) < u_i^a < c_i(a)$, $i = 1, \dots, n$, the corresponding solution $u = u^*(k) = (u_1^*(k), \dots, u_n^*(k))$ of initial problem (1), (2) satisfies the inequalities

$$b_i(k) < u_i^*(k) < c_i(k) \tag{7}$$

for every $k \in \mathbb{Z}_a^\infty$.

2 Application of the general result to equation (3)

Now we will apply Theorem 1 to system (3). Let us denote

$$\omega_i(k) = \frac{\beta_i(k)}{\gamma_i(k)}, \quad i = 1, 2,$$

and

$$f_1(k) = -\frac{\Delta\omega_1(k)}{\omega_2(k)\gamma_1(k)}, \quad f_2(k) = -\frac{\Delta\omega_2(k)}{\omega_1(k)\gamma_2(k)}.$$

Theorem 2 *Let the functions $\gamma_i, \beta_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}^+$, $i = 1, 2$, be given.*
Suppose that for every $k \in \mathbb{Z}_a^\infty$ the following assumptions hold:

1) $\Delta\omega_i(k) < 0$ *for* $i = 1, 2$.

2) $\Delta\omega_i(k) + f_i(k+1) > 0$ *for* $i = 1, 2$.

3) $f_1(k) + f_2(k)(\Delta\omega_1(k))/\omega_2(k) > 0$ *and* $f_2(k) + f_1(k)(\Delta\omega_2(k))/\omega_1(k) > 0$.

4) $\Delta f_i(k) > 0$ *for* $i = 1, 2$.

Then for any initial condition (2) with

$$\omega_i(a) < u_i^a(k) < \omega_i(a) + f_i(a), \quad i = 1, 2,$$

the solution $u = u^*(k) = (u_1^*(k), u_2^*(k))$ of initial problem (3), (2) satisfies the inequalities

$$\omega_i(k) < u_i^*(k) < \omega_i(k) + f_i(k) \tag{8}$$

for $i = 1, 2$ and $k \in \mathbb{Z}_a^\infty$.

Proof. We will apply Theorem 1 with

$$F_1(k,u_1,u_2)=u_2\cdot(\beta_1(k)-\gamma_1(k)u_1),\quad F_2(k,u_1,u_2)=u_1\cdot(\beta_2(k)-\gamma_2(k)u_2),$$

$$b_i(k)=\omega_i(k),\quad c_i(k)=\omega_i(k)+f_i(k),\quad i=1,2.$$

Remark that due to assumption 1), $f_i(k)>0$ for $k\in\mathbb{Z}_a^\infty$, $i=1,2$.

We will prove that all the assumptions of Theorem 1 are satisfied.

The conditions (4) and (5) for $i=1$ in our case become:
If for some $k\in\mathbb{Z}_a^\infty$, $u_1=b_1(k)$ and $b_2(k)\le u_2\le c_2(k)$, then

$$b_1(k+1)<b_1(k)+F_1(k,b_1(k),u_2)<c_1(k+1), \tag{9}$$

and if $u_1=c_1(k)$ and $b_2(k)\le u_2\le c_2(k)$, then

$$b_1(k+1)<c_1(k)+F_1(k,c_1(k),u_2)<c_1(k+1). \tag{10}$$

For $i=2$, the corresponding conditions would be similar. We will prove only the conditions for $i=1$, because due to the symmetry of the studied system, the case $i=2$ is analogous.

Start with inequalities (9). Substituting for $b_1(k)$ and then for $\omega_1(k)$, we get

$$b_1(k)+F_1(k,b_1(k),u_2)=\omega_1(k)+u_2\cdot(-\gamma_1(k)\omega_1(k)+\beta_1(k))=\omega_1(k).$$

Thus, inequalities (9) reduce to

$$\omega_1(k+1)<\omega_1(k)<\omega_1(k+1)+f_1(k+1).$$

The first inequality ($\omega_1(k+1)<\omega_1(k)$) is fulfilled due to assumption 1). The second inequality is equivalent to the inequality

$$\Delta\omega_1(k)+f_1(k+1)>0$$

which is supposed to be valid in assumption 2). Thus, inequalities (9) hold.

Now let us concentrate on inequalities (10). Again, substituting the appropriate values, we get

$$\begin{aligned}c_1(k)+F_1(k,c_1(k),u_2)&=\omega_1(k)+f_1(k)+u_2\cdot(-\gamma_1(k)(\omega_1(k)+f_1(k))+\beta_1(k))=\\&=\omega_1(k)+f_1(k)-u_2\gamma_1(k)f_1(k)=\omega_1(k)+f_1(k)+u_2\frac{\Delta\omega_1(k)}{\omega_2(k)}.\end{aligned}$$

We have to prove that

$$\omega_1(k+1)<\omega_1(k)+f_1(k)+u_2\frac{\Delta\omega_1(k)}{\omega_2(k)}<\omega_1(k+1)+f_1(k+1). \tag{11}$$

Find the lower and the upper estimate of $\omega_1(k)+f_1(k)+u_2(\Delta\omega_1(k))/\omega_2(k)$. In the following considerations we will use the fact that $\Delta\omega_1(k)<0$ (see assumption 1)).

$$\begin{aligned}\omega_1(k)+f_1(k)+u_2\frac{\Delta\omega_1(k)}{\omega_2(k)}&\ge\omega_1(k)+f_1(k)+(\omega_2(k)+f_2(k))\frac{\Delta\omega_1(k)}{\omega_2(k)}=\\&=\omega_1(k)+f_1(k)+\Delta\omega_1(k)+f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}=\omega_1(k+1)+f_1(k)+f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)},\end{aligned}$$

$$\begin{aligned}\omega_1(k)+f_1(k)+u_2\frac{\Delta\omega_1(k)}{\omega_2(k)}&\le\omega_1(k)+f_1(k)+\omega_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}=\\&=\omega_1(k)+f_1(k)+\Delta\omega_1(k)=\omega_1(k+1)+f_1(k).\end{aligned}$$

To prove the first inequality of (11), it is sufficient to prove that

$$\omega_1(k+1) < \omega_1(k+1) + f_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}$$

which gives

$$0 < f_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}.$$

This inequality is fulfilled due to assumption 3) of the Theorem. As for the second inequality from (11), it is sufficient to show that

$$\omega_1(k+1) + f_1(k) < \omega_1(k+1) + f_1(k+1).$$

This holds because of assumption 4). That means that inequalities (10) hold.

As the function

$$G_1(w) := w + F_1(k, w, u_2) = w + u_2 \cdot (-\gamma_1(k)w + \beta_1(k))$$

is linear with respect to its argument w for every fixed $k \in \mathbb{Z}_a^\infty$ and every fixed u_2 such that $b_2(k) \le u_2 \le c_2(k)$, it is monotonous, too. An analogous reasoning could be done also for the function

$$G_2(w) := w + F_2(k, u_1, w) = w + u_1 \cdot (-\gamma_2(k)w + \beta_2(k)).$$

We have shown that all the assumptions of Theorem 1 are fulfilled and thus every solution of system (3) given by an initial condition (2) with $\omega_i(a) < u_i^a < \omega_i(a) + f_i(a)$, $i = 1, 2$, satisfies conditions (8).

Example 1 *Let us consider the system of equations*

$$\begin{aligned} \Delta u_1(k) &= u_2(k)\left(\frac{3}{k} - u_1(k)\right), \\ \Delta u_2(k) &= u_1(k)\left(\frac{1}{k^4} - \frac{1}{k^2}\,u_2(k)\right). \end{aligned} \tag{12}$$

We will show that for $k \in \mathbb{Z}_2^\infty$, all the assumptions of Theorem 2 are fulfilled. In this case we set

$$\omega_1(k) = \frac{3}{k}, \quad \omega_2(k) = \frac{1}{k^2}.$$

Then

$$\Delta\omega_1(k) = \frac{3}{k+1} - \frac{3}{k} = -\frac{3}{k(k+1)}, \quad \Delta\omega_2(k) = \frac{1}{(k+1)^2} - \frac{1}{k^2} = -\frac{2k+1}{k^2(k+1)^2},$$

and

$$\begin{aligned} f_1(k) &= -\frac{-3/(k(k+1))}{(1/k^2)\cdot 1} = \frac{3k}{k+1}, \\ f_2(k) &= -\frac{-(2k+1)/(k^2(k+1)^2)}{(3/k)\cdot(1/k^2)} = \frac{k(2k+1)}{3(k+1)^2}. \end{aligned}$$

Assumption 1) of Theorem 2 is obviously fulfilled.

Prove the validity of assumption 2). First, for $i = 1$:

$$\Delta\omega_1(k) + f_1(k+1) = -\frac{3}{k(k+1)} + \frac{3(k+1)}{k+2} = \frac{3(k^3+2k^2-2)}{k(k+1)(k+2)}$$

The last expression is positive for any $k \in \mathbb{N}$.
Now for $i = 2$:

$$\Delta\omega_2(k) + f_2(k+1) = -\frac{2k+1}{k^2(k+1)^2} + \frac{(k+1)(2k+3)}{3(k+2)^2} = \\ = \frac{2k^6 + 9k^5 + 15k^4 + 5k^3 - 24k^2 - 36k - 12}{3k^2(k+1)^2(k+2)^2}.$$

It can be shown that this expression is positive for $k \in \mathbb{Z}_2^\infty$.
Thus, assumption 2) holds.

The desired inequality in assumption 3) for $i = 1$ is in our case

$$f_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)} = \frac{3k}{k+1} + \frac{k(2k+1)}{3(k+1)^2} \cdot \frac{-3/(k(k+1))}{1/k^2} = \\ = \frac{3k}{k+1} - \frac{k^2(2k+1)}{(k+1)^3} = \frac{k(k^2+5k+3)}{(k+1)^3} > 0.$$

This holds for any $k \in \mathbb{N}$.
For $i = 2$ we get

$$f_2(k) + f_1(k)\frac{\Delta\omega_2(k)}{\omega_1(k)} = \frac{k(2k+1)}{3(k+1)^2} + \frac{3k}{k+1} \cdot \frac{-(2k+1)/(k^2(k+1)^2)}{3/k} = \\ = \frac{k(2k+1)}{3(k+1)^2} - \frac{2k+1}{(k+1)^3} = \frac{2k^3 + 3k^2 - 5k - 3}{3(k+1)^3}$$

which is positive for $k \in \mathbb{Z}_2^\infty$.

As for assumption 4), we get for $i = 1$

$$\Delta f_1(k) = \frac{3(k+1)}{k+2} - \frac{3k}{k+1} = \frac{3}{(k+1)(k+2)} > 0$$

and for $i = 2$

$$\Delta f_2(k) = \frac{(k+1)(2k+3)}{3(k+2)^2} - \frac{k(2k+1)}{3(k+1)^2} = \frac{3k^2+7k+3}{3(k+1)^2(k+2)^2} > 0.$$

All the assumptions of Theorem 2 are fulfilled and thus if there is prescribed an initial condition $u(2) = (u_1^{(2)}, u_2^{(2)})$ such that

$$\frac{3}{2} < u_1^{(2)} < \frac{3}{2} + \frac{3 \cdot 2}{2+1}, \qquad \frac{1}{2^2} < u_2^{(2)} < \frac{1}{2^2} + \frac{2(2 \cdot 2 + 1)}{3(2+1)^2},$$

then the corresponding solution $u = u^*(k)$, $k \in \mathbb{Z}_2^\infty$, of system (12) satisfies the conditions

$$\frac{3}{k} < u_1^*(k) < \frac{3}{k} + \frac{3k}{k+1}, \\ \frac{1}{k^2} < u_2^*(k) < \frac{1}{k^2} + \frac{k(2k+1)}{3(k+1)^2}.$$

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