Sandwich semigroups of solutions of certain functional equations of one variable

Jan Chvalina, Jiří Moučka, Zdeněk Svoboda
Brno University of Technology, Faculty of Electrical Engineering and Communication,
University of Defence, Faculty of economics and management, Brno, Czech Republic
e-mail: chvalina@feec.vutbr.cz, jiri.moucka@unob.cz, svobodaz@feec.vutbr.cz

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Canonical form of linear functional-differential equations are defined by F. Neuman [5]. These special forms are suitable for the investigation of properties of this equations for example oscillatory behavior of all equations from certain classes of linear functional-differential equations because each global pointwise transformation preserves distribution of zeros of solutions of a functional-differential equation and its canonical forms. The most general form of the transformation is

$$z(t) = f(t)y(h(t)), \tag{1}$$

where $f(t) \neq 0$ is continuous and diffeomorphism h has the first derivative $h'(t) \neq 0$. We say that a linear homogeneous functional-differential equation of the first order is transformable on canonical form if there are the function $f(t)$ and diffeomorphism $h(t)$ (with requisite properties) such that holds: any function $y(t)$ is solution if and only if the function $z(t)$ is solution of equations bellow:

$$y'(x) + \sum_{i=0}^{m} p_i(x)y(\xi_i(x)) = 0, \quad z'(t) + \sum_{i=0}^{m} q_i(t)z(t + c_i) = 0, \tag{2}$$

If the equations above are transformable then difeomorphism h satisfies the equation:

$$\xi_i(h(t)) = h(t + c_i), \quad \text{for } i = 1, \ldots, n \tag{3}$$

Moreover for deviations $\xi_i(x)$ hold

$$\xi_i \circ \xi_j = \xi_j \circ \xi_i, \tag{4}$$

From the above follows that the investigations of neutrally commuting systems of functions belong to important questions. If we investigate solvability and describing the solution we use the Cayley graphs of functions which are contained in equations. Using this method we may for example say that the equation

$$a^{f(x)} - f(ax(a + |x|)^{-1}) = 0, \tag{5}$$

has no solution for $x \in \mathbb{R}$ but for the restriction of functions which are in the equations onto the set $R - \{0\}$ the set of solution of this equations has the cardinality of continuum.

Moreover there is studied the algebraic structure of solution sets of these equations. In monograph [2] chapt. I, 3 p-morphisms are called strongly isotone mappings or strong homomorphisms.From the results of chapter II and analogical analysis contained in the 1 of the chapter III of the book [2] we get the following theorem.

Theorem 1 *Suppose that $a > 0$. There exists an infinit set of order relations $\{\leq_\alpha; \alpha \in \mathbb{N}\}$ of the set $\mathbb{R}$ of all real numbers with following property:*
For any $\alpha \in \mathbb{N}$ a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of a functional equation

$$af(x) = f(ax(a^2 + x^2)^{-\frac{1}{2}})\sqrt{[f(x)]^2 + a^2},$$

if and only if the function f is strongly isotone selfmapping of the ordered set $(\mathbb{R}, \leq_\alpha)$.

For any element a of a semigroup S we may define a "sandwich" operation $\cdot$ on the set S by $x \cdot y = xay$, $x, y \in S$. Under this operation the set S is again a semigroup; it is denoted by (S, a) and called a variant of S. A certain generalization are sandwich semigroups investigated in papers of Kenneth D. Magill. We note that the Cayley graphs of functions $\varphi_a(x) = \frac{ax}{\sqrt{x^2+a^2}}$ and $\psi_b(x) = \frac{bx}{|x|+b}$ are isomorphic. Consider the functional equations of one real variable with real parameters a, b.

$$f(\varphi_a(x)) = \psi_b(f(x)), \quad x \in \mathbb{R} \tag{6}$$

in the above notation. Similarly, the equation (7) leads the equation

$$g(\psi_b(x)) = \varphi_a(g(x)), \quad x \in \mathbb{R}. \tag{7}$$

Let $g_0 : \mathbb{R} \to \mathbb{R}$ be an arbitrary surjective solution of the functional equation (7).Denote by $S(\varphi_a, \psi_b)$ the solution set of the equation (6) in which we define the following binary operation:

For an arbitrary pair $f_1, f_2 \in S(\varphi_a, \psi_b)$ we put $f_1 \cdot f_2 = f_2 \circ g_0 \circ f_1$. From the above mentioned results contained in the monograph [2] there follows the following result:

Proposition. *There exists a pair of orderings $\leq_1, \leq_2$ of the set $\mathbb{R}$ of all real numbers such that the sandwich semigroups $S(\leq_1, \leq_2, g_0)$ of all strongly isotone functions $f : (\mathbb{R}, \leq_1) \to (\mathbb{R}, \leq_2)$ with the sandwich function $g_0 : \mathbb{R} \to \mathbb{R}$ which is a strongly isotone mapping $g_0 : (\mathbb{R}, \leq_1) \to (\mathbb{R}, \leq_2)$, coincide with the sandwich semigroup $S(\varphi_a, \psi_b, g_0)$.*, i.e. $S(\varphi_a, \psi_b, g_0) = S(\leq_1, \leq_2, g_0)$.

Notice that the above presentation is very brief – it contains only basic ideas. Therefore investigation of properties of the just constructed sandwich semigroups and further development

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