Sandwich semigroups of solutions of certain functional equations of one variable

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Abstract

Investigation of canonical forms of a certain class of functional differential equations needs treatment of two type equations $h \circ \varphi = \psi \circ h$ and $h \circ \varphi = \varphi \circ h$. For respond to solvability and describing the solution we use the Cayley graphs of functions which are contained in equations. This method is illustrated on examples. Moreover there is studied the algebraic structure of solution sets of these equations.

1 Introduction

Canonical form of linear functional-differential equations are defined by F. Neuman [13]. This form is established for the classes of equations which is possible pointwise transform on equation with constant deviation. These special forms are suitable for the investigation of properties of this equations for example oscillatory behavior of all equations from certain classes of linear functional-differential equations because each global pointwise transformation preserves distribution of zeros of solutions of a functional-differential equation and its canonical forms. The most general form of the transformation is $z(t) = f(t)y(h(t))$, where $f(t) \neq 0$ is continuous and diffeomorphism h has the first derivative $h'(t) \neq 0$.

We say that the pointwise transformation of a linear homogeneous functional-differential equation of the first order on canonical form means that the functions $y(x) \in C(\mathcal{X})$ is solution of equation

$$y'(x) + \sum_{i=0}^{m} p_i(x)y(\xi_i(x)) = 0, \tag{1}$$

where $x \in \mathcal{X}$ and $\mathcal{X} \subseteq \mathbb{R}$ is any righthand open interval,
if and only if
the function $z(t) \in C(\mathcal{T})$ is solution of equation:

$$z'(t) + \sum_{i=0}^{m} q_i(t)z(t + c_i) = 0, \tag{2}$$

with constants c_i where $t \in \mathcal{T}$ and $\mathcal{T} \subseteq \mathbb{R}$ is any interval. It means that there is a function $f \in C^1(\mathcal{T})$, which $f(t) \neq 0$ on $\mathcal{T}$ and diffeomorphism h $\mathcal{T}$ onto $\mathcal{X}$ i.e. $h \in C^1(\mathcal{T})$ such that $h(\mathcal{T}) = \mathcal{X}$, $h'(t) \neq 0$ on $\mathcal{T}$ and

$$\xi_i(h(t)) = h(t + c_i), \quad \text{for } i = 1, \dots, n \tag{3}$$

$$z(t) = f(t)y(h(t)), \tag{4}$$

$y(t)$ is solution of equation (1) whenever $z(t)$ is solution (2). The situation is described in [22]. In this paper is formulated the theorem which we cite here.

Theorem 1 *Suppose that an equation (1) is globally transformable into an equation (2) and there exist two solutions $y_1, y_2 \in C^1(I)$ of (1) with the nonzero Wronskian determinant, $p_k \neq 0$ on I for some $k \in \{1, 2, \ldots, m\}$. Then (2) is an equation with constant coefficients and constant deviations if and only if for function $\xi \in \{\xi_1, \xi_2, \ldots, \xi_m\}$ there exists a function $L : I \to R$, $L \in C^1(I), L(x) \neq 0$ on I such that the relations*

$$L'(x)/L(r) = p_0(x) - p_0(\xi(x))\xi'(x),\ \ p_i(\xi(x))\xi'(x)L(\xi_i(x)L(x) = p_i(x), i \in \{1, 2 \ldots, m\}, \quad (5)$$

are satisfied on I. Moreover, $q_0(t) \equiv 0$ on $J = [a_1, \infty)$, $J = (-\infty, \infty)$ respectively.

Moreover for deviations $\xi_i(x)$ hold

$$\xi_i \circ \xi_j = \xi_j \circ \xi_i, \quad (6)$$

for this result see [15].

From the above follows that the investigations of neutrally commuting systems of functions belong to important questions. There are known some algebraic constructions which allow to us construction of solutions set or solution monoids of certain functional equations of one real variable from special classes such equations.

2 Preliminaries

If X is an arbitrary set and $\xi\ : X \to X$ a mapping, the pair (X, ξ) is called a mono-unary algebra. The $n - th$ iteration of ξ is defined for $n \in \mathbb{N}$ by $\xi^1 = \xi$, $\xi^{n+1} = \xi \circ \xi^n$, where $\circ$ stands for the binary operation of composition of functions; ξ^0 is the identity mapping, i.e. $\xi^0 = id_x$. Any self-map $\xi\ : X \to X$ determines the Kuratowski Whybourn equivalence $\sim_\xi$ on X (or KW-equivalence) defined by $x, y \in X$,

$$x\ \sim_\xi\ y \Leftrightarrow \xi^n(x) = \xi^m(y) \text{ for some pair } m, n \in \mathbb{N} \cup \{0\}.$$

Blocks of corresponding partition $X/\sim_\xi$ are called orbits of the mapping ξ or $\xi-$ orbits. Subalgebras $(Y, \xi|Y)$ of the mono-unary algebra (X, ξ), where Y is an orbit, are called components of the algebra (X, ξ). Components of (X, ξ) are maximal connected subalgebras of (X, ξ); here the algebra (X, ξ) is said to be connected if $x\ \sim_\xi\ y$ for any pair of its elements x, y and an algebra (Y, η) is a subalgebra of (X, ξ) if $Y \subseteq X$ and η is a restriction of the mapping ξ onto Y, i.e. $\eta = \xi|Y$. Therefore, if $\{(Y_r, \eta_r); r \in I\}$ is the system of all components of a mono-unary algebra (X, ξ), we write (with respect to a cardinal sum of mono-relational structures)

$$(X, \xi) = \sum_{r \in I} (Y_r, \eta_r).$$

The graphic form of the above expression is also called the Cayley graph of the considered function. The following two examples of connected mono-unary algebras are $(\mathbb{Z}, \nu)$, $(\mathbb{N}, \nu)$, where $\mathbb{Z}$ is set of all integer and $\mathbb{N}$ its subset of all positive integers and $\nu(n) = n + 1$. Let us also remind that functions g, h are said to be conjugated if for some bijection $\varphi : R \to R$ we have $\varphi(g(x)) = h(\varphi(x))$ for any $x \in \mathbb{R}$. This means that $\varphi : (R, g) \to (R, h)$ is an isomorphism of mono-unary algebras (R, g), (R, h). We denote conjugacy of functions g, h as $g \backsimeq h$, or more precisely $(R, g) \backsimeq (R, h)$. By the iterated sequence or splinter (or rather ξ-splinter) generated by the element x_0 we mean a sequence $\xi^n(x_0)_{n \in N_0}$, where $\xi : R \to R$ is a function and $x_0 \in R$. Finally $S(R)$ denotes the group of all bijections of R onto itself. As usual, by a monoid we mean a semihypergroup with a unit.

3 Results

Now we consider the functions which are suitable for demonstration of algebraic construction. This construction operate with Cayley graph of these function and $n-th$ iteration of these.

$$\varphi_a(x) = \frac{ax}{a+|x|}, \quad \text{where} \ \ a>0, \ \ x\in\mathbb{R} \tag{7}$$

$$\psi_a(x) = \frac{ax}{\sqrt{x^2+a^2}}, \quad \text{where} \ \ a>0, \ \ x\in\mathbb{R} \tag{8}$$

$$y = x^{2k+1},, \quad \text{where} \ \ k\in\mathbb{N} \; x\in R \tag{9}$$

$$y = a^x, \quad \text{where} \ \ a>0, \ \ x\in\mathbb{R} \tag{10}$$

For this functions we generate these.

Function $\varphi_a(x)$ has the $n-th$ iteration $\varphi_a^n(x) = \dfrac{ax}{a+n\,|x|}$ and orbit structure of this function has form

$$(\mathbb{R},\varphi_a) = \{0\} \quad + \sum_{\alpha\in(0,1)} (X_\alpha,\xi_\alpha),$$

where $\xi_\alpha = \varphi_a|X_\alpha$, $\alpha\in(0,1)$ a $(X_\alpha,\xi_\alpha)\cong(\mathbb{N},\nu)$.

Analogically we have that function $\psi_a(x)$ has the $n-th$ iteration $\psi_a^n(x) = \dfrac{ax}{\sqrt{a^2+nx^2}}$ and orbit structure of this function has the isomorphic form with the function $\varphi_a(x)$

$$(\mathbb{R},\psi_a) = \{0\} \quad + \sum_{\alpha\in(0,1)} (X_\alpha,\xi_\alpha),$$

where $\xi_\alpha = \varphi_a|X_\alpha$, $\alpha\in(0,1)$ a $(X_\alpha,\xi_\alpha)\cong(\mathbb{N},\nu)$.

We note that these formulae for the functions $\varphi_a^n(x)$, $\psi_a^n(x)$ hold for all integer n, if for $n=-1$ denotes inverse function and for $-n$, where n is positive denotes of the $n-th$ iteration of inverse function. For $n=0$ the formulae give the identity.

For the function $q_k(x) = x^{2k+1}$ we obtain $n-th$ iteration $q_k^n(x) = x^{n(2k+1)}$ and orbit structure of this function has form

$$(\mathbb{R},q_k) = \{-1\} \quad +\{0\} \quad +\{1\} \quad + \sum_{\alpha\in(0,1)} (X_\alpha,\zeta_\alpha),$$

where $\zeta_\alpha = p_a|X_\alpha$, $\alpha\in(0,1)$ a $(X_\alpha,\zeta_\alpha)\cong(\mathbb{Z},\nu)$ for every element $(0,1)$.

Finally for the function $p_a(x) = a^x$ we have $n-th$ iteration

$$p_a^n(x) = \underbrace{\ln a\exp(\ldots\ln a\exp(\ln a\exp(x))\ldots)}_{n\text{times}}$$

and orbit structure of this function has form

$$(\mathbb{R},p_a) = \sum_{\alpha\in(0,1)} (X_\alpha,\xi_\alpha),$$

where $\xi_\alpha = p_a|X_\alpha$, $\alpha\in(0,1)$ a $(X_\alpha,\xi_\alpha)\cong(\mathbb{N},\nu)$ for every element $\alpha\in$ (0,1).

3.1 Solving of equations

Now we use the orbit structure of the function $\psi_a(x)$ for specification of the set all solutions of equation

$$af(x) - (a + |f(x)|)f(ax(a + |x|)^{-1}) = 0, \tag{11}$$

which is possible to rewrite in the form so called centralized equation

$$f\left(\frac{ax}{a+|x|}\right) = \frac{af(x)}{a+|f(x)|} \equiv f \circ \varphi_a = \varphi_a \circ f$$

For $x_0 \in G = (-\infty, -a\rangle \cup \langle a, \infty)$ there holds $\varphi_a^{-1}(x_0) = \emptyset$. We use the orbit structure

$$(\mathbb{R}, \varphi_a) = \{0\} \quad + \sum_{\alpha \in (0,1)} (X_\alpha, \xi_\alpha),$$

for the construction of all solution f of the equation (11). For every function $\gamma : \quad G \to \mathbb{R}$ with property $\gamma(0) = 0$ we construct function f_γ:

For $x = 0$ we put $f_\gamma(0) = 0$. Let $x \in \mathbb{R} - \{0\}$, then there is just only one pair of numbers $[x_0, n]$ where n is nonnegative integer and $x_0 \in G - \{0\}$ with property

$$x = \varphi_a^n(x_0) = \frac{ax_0}{a + n|x_0|}.$$

Then we put

$$f_\gamma(x) = \varphi_a^n \gamma(x_0) = \frac{a\gamma(x_0)}{n|\gamma(x_0)| + a}.$$

Using this construction we define for $x \in \mathbb{R}$ the function $f_\gamma : \quad \mathbb{R} \to \mathbb{R}$, which satisfies:

$$f_\gamma(ax(|x| + a)^{-1}) = f_\gamma(\varphi_a(x)) = f_\gamma(\varphi_a^{n+1}(x_0)) = \frac{a\gamma(x_0)}{(n+1)|\gamma(x_0)| + a} = \varphi_a\left(\frac{a\gamma(x_0)}{n|\gamma(x_0)| + a}\right) = \varphi_a(f_\gamma(x))$$

for every $x \in \mathbb{R}$. Consequently the function f_γ is solution of the the functional equation $f(ax(|x| + a)^{-1}) = af(x)(|f(x)| + a)^{-1}$, it is equation (11).

Using the analogous consideration we may discussed the equations

$$af(a^x) + |f(x)|f(a^x) = af(x), \tag{12}$$

$$a^{f(x)} - f(ax(a + |x|)^{-1}) = 0, \tag{13}$$

$$af(x^{2n+1}) + |f(x)|f(x^{2n+1}) = af(x), \tag{14}$$

$$(f(x))^{2n+1} = f(ax(a + |x|)^{-1}) \tag{15}$$

$$(f(x))^{2n+1} - f(a^x) = 0 \tag{16}$$

$$f(x^{2n+1}) - a^{f(x)} = 0 \tag{17}$$

$$a^{f(x)} - f(ax(x^2 + a^2)^{-1/2}) = 0 \tag{18}$$

$$f(a^x)\sqrt{(f(x))^2 + a^2} = af(x) \tag{19}$$

$$\sqrt{(f(x))^2 + a^2} \cdot f(ax(|x| + a)^{-1}) = af(x) \tag{20}$$

$$(|f(x)| + a)f(ax(a^2 + x^2)^{-1/2}) = af(x) \tag{21}$$

$$(f(x))^{2n-1} - f(ax(a^2 + x^2)^{-1/2}) = 0 \tag{22}$$

$$f(x^{2n+1}) \cdot \sqrt{a^2 + (f(x))^2} = af(x) \tag{23}$$

Since for components which are isomorphic with algebra (Z, ν_z) do not exist homomorphisms onto components which are isomorphic with algebra (N, ν) and there is no homomorphism which maps the simple loop in the infinite components, the functional equations (13), (17), (18) has no solution and equations (14) and (23) has just one trivial solution. The other equation have the set of solutions with the cardinality of continuum.We note that for the restriction of functions which are in the equations (13) and (18) onto the set $R-\{0\}$ the set of solution of these equations has the cardinality of continuum.

If we use this construction of solution the consideration of useful properties of this solution (continuity..) is difficult. For formulation sufficient condition which assure continuity we need specification of the set of generators G and the n-iterated function. For the solved equation (11) we have

$$G^0 = (-\infty, -a\rangle \cup \langle a, \infty)$$

and we denote

$$G^n = \varphi_a^n(G) = \left(-\frac{a}{n}, -\frac{a}{n+1}\right\rangle \cup \left\langle \frac{a}{n+1}, \frac{a}{n}\right).$$

The continuity of solution in interior of interval is equivalent with the continuity on the intervals G^0 i.e. with continuity of function γ. Continuity of solution in the point of the boundary of intervals is equivalent the condition

$$f_\gamma\left(-\frac{a}{n}\right) = \lim_{x\to -\frac{a}{n}+} f_\gamma(x) \equiv \frac{a\gamma(-a)}{a + n|\gamma(-a)|} = \lim_{x\to-\infty} \frac{a\gamma(x)}{a + (n+1)|\gamma(x)|}.$$

Analogous relation holds for $f_\gamma(a)$ and $f_\gamma(\infty)$. Now we may formulate the assertion.

Assertion. *The function $f(x)$ is continuous solution of the equation (11) if and only if the restriction γ of the function f onto set of generators G has finite limits $\gamma(\pm\infty) = \lim_{x\to\pm\infty} \gamma(x)$ and there hold the relations*

$$\gamma(-\infty)(a + \gamma(-a)(n \operatorname{sign}(\gamma(-a)) - (n+1)\operatorname{sign}(\gamma(-\infty)))) = -a\gamma(-a) \tag{24}$$

$$\gamma(\infty)(a + \gamma(a)(n \operatorname{sign}(\gamma(a)) - (n+1)\operatorname{sign}(\gamma(\infty)))) = -a\gamma(a) \tag{25}$$

3.2 Logics and ordered sets

Kripke semantics (also known as relation semantics or frame semantics) is a formal semantics for non–classical logic systems created in the late 1950s and early 1960s by Saul Aaron Kripke. A Kripke frame or modal frame is a pair $< W, R >$, where W is a non–empty set, and R is a binary relation on W. Elements of W are called nodes or worlds, and the relation R is known as the accessibility relation. This is a binary relation between possible worlds which has very powerful uses in both the formal/theoretic aspects of modal logic as well as in its applications to thinks like epistemology and value theory. As in the classical model theory, there are methods for constructing a new Kripke model from other models.

The natural homomorphisms in Kripke semantics are called p-morphisms (or pseudo–epimorphisms, but the latter term is rarely used). A p-morphism of Kripke frames $< W, R >$ and $< W', R' >$ is a mapping $f : W \to W'$ such that f preserves the accessibility relation, i.e. xRy implies $f(x)R'f(x)$, and whenever $f(x)R'y'$ there is a node $y \in W$ such that xRy and $f(y) = y'$. Notice that p-morphisms are special kind of so called bisimulations.

In monograph [5] chapt. I, 3 p-morphisms are called strongly isotone mappings or strong homomophisms and such mapping can be characterized ([5], Proposition 3.3) as mapping satisfying the condition: For any $x\imath W$, there holds $R'(f(x)) = f(R(x))$ (where $R(x) = \{y|xRy\}$). In words - the image of principal R-end generating by the image $f(x)$. If we consider strong endomorphisms $f : (W,R) \to (W,R)$, we will denote the monoid of all such mapping by Send(W,R) or Send(A,p) in our notation.

From the results of chapter II and analogical analysis contained in the 1 of the chapter III of the book [5] we get the following theorem.

Theorem 2 *Suppose that $a > 0$. There exists an infinit set of order relations $\{\leq_\alpha; \alpha \in \mathbb{N}\}$ of the set $\mathbb{R}$ of all real numbers with following property:*
For any $\alpha \in \mathbb{N}$ a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of a functional equation

$$af(x) = f(ax(a^2 + x^2)^{-\frac{1}{2}})\sqrt{[f(x)]^2 + a^2},$$

if and only if the function f is strongly isotone selfmapping of the ordered set $(\mathbb{R}, \leq_\alpha)$.

3.3 Sandwich semigroups

In the case of centralizer functional equations of one variable it is natural the solution sets endow with an appropriate algebraic structure. In fact a centralizer of a function φ within the ring of all real functions is the monoid of all functions $f : \mathbb{R} \to \mathbb{R}$ commuting with φ. In the case of solution sets of functional equations of the form

$$f \circ \varphi_1 = \varphi_2 \circ f,$$

i.e. $f(\varphi(x)) = \varphi(f(x))$, $x \in J$, $J \subseteq \mathbb{R}$, where $\varphi_1, \varphi_2 \quad J \to \mathbb{R}$ are given different functions we have another situation. In order to define binary structure on solution sets of the just mention equations which is based on the binary operation of composition of function we are concerning to the concept of a sandwich semigroup. These semigroups were investigated since the end of sixtieth (by Kenneth D.Magill Jr. - 1967) of the last century; let us mentioned at least papers [9, 11, 12] and recall the basic notion.

According to paper [9] for any element a of a semigroup S we may define a "sandwich" operation $\cdot$ on the set S by $x \cdot y = xay$, $x, y \in S$. Under this operation the set S is again a semigroup; it is denoted by (S, a) and called a variant of S (in [9]). A certain generalization of a variant suitable for our purposes are sandwich semigroups investigated in papers of Kenneth D. Magill, Jr. and his collaborates.

Consider the following two functional equations of one real variable with two real parameters a, b.

$$f\left(ax(x^2 + a^2)^{-\frac{1}{2}}\right)(|f(x) + b) = bf(x) \tag{26}$$

and

$$g(bx(|x| + b)^{-1})\sqrt{[g(x)]^2 + a^2} = ag(x). \tag{27}$$

Then the parameters a, b coincide (i.e. $a = b$) the the equations (26), (27) turn out in the above considered equations (21), (20), respectively. From the equality (26) we get without any any effort

$$f\left(\frac{ax}{\sqrt{x^2 + a^2}}\right) = \frac{bf(x)}{|f(x)| + b},$$

i.e.

$$f(\varphi_a(x)) = \psi_b(f(x)), \quad x \in \mathbb{R} \tag{(26a)}$$

in the above notation. Similarly, the equation (27) leads the equation

$$g(\psi_b(x)) = \varphi_a(g(x)), \quad x \in \mathbb{R}. \tag{(27a)}$$

Let $g_0 : \mathbb{R} \to \mathbb{R}$ be an arbitrary solution of the functional equation (27, 27a) which is surjective, i.e. $g_0(\mathbb{R}) = \mathbb{R}$ it has a fixed point 0, i.e. $g_0(0) = 0$. Denote by $S(\varphi_a, \psi_b)$ the solution set of the equation (26) in which we define the following binary operation:

For an arbitrary pair $f_1, f_2 \in S(\varphi_a, \psi_b, g_0)$ we put $f_1 \cdot f_2 = f_2 \circ g_0 \circ f_1$. Then we obtain a sandwich semigroup will be denoted by $S(\varphi_a, \psi_b, g_0)$. Evidently $f_1 \cdot f_2 \in S(\varphi_a, \psi_b, g_0)$ of the functional equation (26, 26a).

From the above mentioned results contained in the monograph [5] there follows the following result:

Proposition. *There exists a pair of orderings $\leq_1, \leq_2$ of the set $\mathbb{R}$ of all real numbers such that the sandwich semigroups $S(\leq_1, \leq_2, g_0)$ of all strongly isotone functions $f : (\mathbb{R}, \leq_1) \to (\mathbb{R}, \leq_2)$ with the sandwich function $g_0 : \mathbb{R} \to \mathbb{R}$ which is a strongly isotone mapping $g_0 : (\mathbb{R}, \leq_1) \to (\mathbb{R}, \leq_2)$, coincide with the sandwich semigroup $S(\varphi_a, \psi_b, g_0)$.*, i.e. $S(\varphi_a, \psi_b, g_0) = S(\leq_1, \leq_2, g_0)$.

Professor M. C. Zdun solved in [23] a generalization of the problem studied by F. Neuman in [15] concerning topological properties of the sets $\{(x, f^n \circ g^m(x));\ m, n \in \mathbb{Z}, x \in I\}$ in the case when f, g are commuting functions which are continuous invertible self–mappings of an interval $I = (a, b)$, $-\infty \leq a < b \leq \infty$ belonging to the same iteration group. M. C. Zdun supposes moreover that $f^n(x) \neq g^m(x)$ for every point $x \in (a, b)$ and every pair integers $m, n \in \mathbb{Z}$ which are not equal to zero simultaneously. Under mentioned hypotheses he obtained that for any pair of points $x, y \in (a, b)$ the set $L(x)$, $L(y)$ of the sequences $\{f^n \circ f^m(x); n, m \in \mathbb{Z}\}$ and $\{f^n \circ f^m(y); n, m \in \mathbb{Z}\}$ coincide. Denoting $L(x) = L$ (for any $x \in (a, b)$) there is proved in [23] that the set $L \cap (a, b)$ is f–invariant and g–invariant and L is either perfect (dense in itself and closed) or $L =< a, b >$. Further properties of the set L are derived. Moreover in [23] there is also proved that isf $f, g\ : (a, b) \to (a, b)$ are strictly increasing commuting bijections with the property $f^n = g^m$ for some pair of integers n, m then there exists a bijection $h\ : (a, b) \to (a, b)$ and integers p, q such that $f = h^q$, $g = h^p$ in the second paper [24] there are investigated continuous and measurable solutions of the system of Abel's functional equations

$$\varphi(f(x)) = \varphi(x) + 1, \quad \varphi(g(x)) = \varphi(x) + s, \quad x \in (a, b)$$

under the supposition that f, g are commuting continuous permutations of the interval (a, b) satisfying the above mentioned condition $f^n \neq g^m$ for any pair $n, m \in \mathbb{Z}$ with $|n| + |m| \neq 0$. The goal of the just described approach are results of topological and analytical character based on the use of the natural topology of the real line, whereas our approach is rather discrete–algebraical. Nevertheless organic combination of both directions of investigation can be leading to new interesting results.

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