Symmetries of differential equations

Veronika Chrastinová
Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics
e-mail: chrastinova.v@fce.vutbr.cz

The common approach to symmetries

Our idea will be illustrated on the ordinary differential equation

$$f(x, y, y', \cdots, y^{(r)}, z, z', \cdots, z^{(r)}) = 0. \qquad (\star)$$

In the classical approach, the equation is considered in the space of jet variables

$$x, y, y', \cdots, y^{(r)}, z, z', \cdots, z^{(r)}.$$

Then, due to the Lie-Bäcklund theorem, *a group acting in this space consists of mere point transformations*

$$\bar{x} = g(\lambda; x, y, z),\ \bar{y} = h(\lambda; x, y, z),\ \bar{z} = k(\lambda; x, y, z) \quad \text{where } \lambda \in \mathbb{R}.$$

In appropriate coordinates, equations of this group simplify as

$$\bar{x} = x + \lambda,\ \bar{y} = y,\ \bar{z} = z. \qquad (\text{hence } \overline{y^{(s)}} = y^{(s)},\ \overline{z^{(s)}} = z^{(s)}).$$

In these coordinates, all functions (invariants of the group)

$$F(y, y', \cdots, y^{(r)}, z, z', \cdots, z^{(r)})$$

are preserved and it follows that *the formula*

$$f = GF = 0 \qquad (G \neq 0)$$

with arbitrary G represents all differential equations symmetrical with respect to this group. The equation $f = 0$ is expressed in terms of new coordinates, including the invariants of the group.

The converse problem when the equation ($\star$) is given in terms of general coordinates and we search for the symmetry group in order to obtain the representation in terms of invariants is more complicated and treated in many textbooks.

Non-classical symmetries

However, the classical theory does not solve the symmetry problem completely. There exist transformation groups of quite other kind, for instance the group

$$\bar{x} = x,\ \bar{y} = y,\ \bar{z} = z + \lambda y' \text{ where } \lambda \in \mathbb{R},$$

which do not preserve any jet space and therefore *cannot be included into the classical theory.* The invariants can be found, namely $\overline{y^{(s)}} = y^{(s)}$ and moreover

$$\overline{z/y'} = \bar{z}/\bar{y}' = z/y' + \lambda,\ \overline{(z/y)'} = (z/y')', \cdots \quad .$$

It follows that all the functions

$$F(x,y,y',\cdots,y^{(r)},z/y',(z/y')',\cdots,(z/y')^{(r)})$$

are invariants and therefore *differential equations*

$$GF = 0 \qquad (G \neq 0)$$

with arbitrary G are preserved if the group is applied. The classical methods are insufficient to determine the symmetry group in this case.
There is an unimaginable amount of such groups which do not preserve the jet spaces and therefore many differential equations where the classical menthods fail. They are however more involved.
Let us deal with the group

$$\bar{x} = x + \lambda\,\frac{y'}{z'},\;\; \bar{y} = y,\;\; \bar{z} = z \qquad (\lambda \in \mathbb{R}, z' \neq 0).$$

Then the derivatives are transformed by rather clumsy recurrence formulae

$$\overline{y^{(s+1)}} = D\,\overline{y^{(s)}}/\,D\bar{x},\;\; \overline{z^{(s+1)}} = D\,\overline{z^{(s)}}/\,D\bar{x}$$

where

$$D = \frac{\partial}{\partial x} + \sum y^{(s+1)}\,\frac{\partial}{\partial y^{(s)}} + \sum z^{(s+1)}\,\frac{\partial}{\partial z^{(s)}}$$

and direct calculation of all invariants is possible but not easy.
Fortunately, the alternative method can be applied in this particular case. If z is chosen for a new independent variable (instead of x), we obtain the group

$$\bar{x} = x + \lambda/\dot{y},\;\; \bar{y} = y,\;\; \bar{z} = z \qquad (. = \mathrm{d}/\mathrm{d}z)$$

by applying $z' = 1/\dot{x}$, $y' = \dot{z}/\dot{x}$. Therefore $\overline{\dot{y}} = \dot{y}$, $\overline{\ddot{y}} = \ddot{y}$, $\ldots$ and moreover

$$\overline{x\dot{y}} = \bar{x}\,\overline{\dot{y}} = x\dot{y} + \lambda,\;\; \overline{(x\dot{y})^{\cdot}} = (x\dot{y})^{\cdot},\;\ldots\;.$$

It follows that all the functions

$$F(z,y,\dot{y},\cdots,y^{(r)},(x\dot{y})^{\cdot},\cdots,(x\dot{y})^{(r)})$$

are invariants and therefore we have differential equations $GF = 0 \quad (G \neq 0)$ with the above symmetry group. Turning to the original variables, we obtain many invariant equations which need not be stated here. For instance the simple equation

$$G(\cdots)\,F(z,y,\dot{y}) = G(\cdots)\,F(z,y,y'/z') = 0$$

with arbitrary G where the symmetry group again cannot be determined by the classical method.

A universal method for determining *all* the symmetries of differential equations does not exist yet.

References

[1] V. Chrastinová, V. Tryhuk: *Automorphisms of curves* (to appear)

[2] P. J. Olver: *Applications of Lie groups to differential equations,* Graduate Texts in Mathematics 107, Springer Verlag 1989.