On an application of a recovered gradient technique

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Abstract

A method for improving the accuracy of the gradient of a finite element solution to a boundary value problem (BVP) is recalled. It is proposed to use the resulting recovered gradient in sensitivity analysis formulas that appear in minimization procedures stemming from BVP-based worst scenario or parameter identification problems, for example. Although the recovered gradient does not help to directly accelerate the minimization process, it can be used as an indicator whether or not the minimum found is sufficiently accurate.

1 Introduction

Some kinds of optimization problems stem form parameter-dependent BVPs whose solution is evaluated through a criterion functional (cost functional). Then, the goal is to find the parameter value that minimizes the functional. Illustrative problems can be found in the worst scenario method, parameter optimization, or parameter identification.

In mathematical language, we search for

$$a_0 = \arg\max_{a \in \mathcal{U}_{\mathrm{ad}}} \Psi(a), \tag{1}$$

where $\mathcal{U}_{\mathrm{ad}}$ is a set of admissible parameters and Ψ is a criterion functional defined as $\Psi(a) = \Phi(a, u(a))$, which is possible if $u(a)$ is the unique solution to the underlying a-dependent BVP.

To find at least an approximate solution to (1), one has to resort to (a) a numerical tool for solving the BVP (take the finite element method (FEM), for instance), and (b) a finite-dimensional approximation of $\mathcal{U}_{\mathrm{ad}}$. The search for an approximation of the minimizer a_0 is then formulated as a constrained minimization of Ψ_h, a (usually nonlinear) function of several variables.

2 Sensitivity Analysis

If the minimization problem (1) and its approximation allow for the differentiation of Ψ with respect to a and a_h (a finite-dimensional approximation of a), then the knowledge of $\nabla\Psi_h(a_h)$, of the gradient of Ψ_h with respect to the components of a_h, pays off in the minimization process.

To obtain $\nabla\Psi_h(a_h)$, different methods can be applied, see [1]; let us stick to formulas originating from the weak formulation of the BVP. In these formulas, the gradient of the (approximate) state solution and the gradient of the related adjoint solution appear. A question arises whether or not the minimization procedure can be made more efficient by improving the accuracy of the gradients involved.

3 Gradient Recovery

There are a number of techniques for improving the accuracy of the gradient of a FEM solution to a BVP. We will consider the method proposed in [2] that is based on a rather simple averaging technique and can be applied to piecewise linear continuous functions in 2D or 3D. It is proved that under certain assumptions and by using this technique, we can get the order of convergence equal to h^2 instead of the common order h, where h is the mesh size parameter. It is natural both to apply the averaged gradient technique in postprocessing the gradients outputting from the FEM and to use the recovered gradients to the sensitivity evaluation.

4 Conclusions

The above idea was tested on an elliptic second order BVP with the criterion functional that evaluates the difference between the state solution and a fixed function in the squared L^2-norm. The parameter a stands for a polynomial coefficient in the partial differential equation that (together with homogeneous Dirichlet boundary conditions) defines the BVP.

It turns out that the replacement of approximate gradients by their recovered counterparts is worthwhile in the right-hand side of the adjoint equation if the corresponding criterion functional depends on the gradient of the state solution. In this case, the difference between the minimum value obtained without the recovered gradient and the minimum value obtained through the recovered gradient indicates whether or not the approximations of a_0 and $\Psi(a_0)$ are sufficiently accurate.

However, as can be expected, the recovered state and adjoint gradients substituted directly into the sensitivity formula do not profitably contribute to the minimization process because the sensitivity formula is inferred for the original approximate gradients and the use of the recovered gradients decreases its accuracy.

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References

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