Linear stability analysis of viscous flow in axisymmetric domain

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Abstract

The linear stability problem of an incompressible viscous flow between two concentric cylinders is investigated. Linearizing the transient behavior around a steady state solution leads to an eigenvalue problem for linearized Navier-Stokes equations. The discrete eigenvalue problem is obtained by the spectral element method. The algorithm is implemented in MATLAB. The developed program serves as a simple tool for numerical experimenting. The applicability of the program is validated through some few test computations.

1. Flow equations

Let $D = \{(r,z)\,|\,r > 0\}$ is a 2D domain with the boundary $\partial D \equiv \Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Through the rotation of D about the z-axis we arrive at the axisymmetric 3D domain

$$V = \{(x,y,z)\,|\,x = r\cos\varphi, y = r\sin\varphi\,, (r,z) \in D, \varphi \in \langle 0, 2\pi)\}$$

with the boundary $\partial V \equiv S = \bar{S}_1 \cup \bar{S}_2, S_1 \cap S_2 = \emptyset$, where

$$S_i = \{(x,y,z)\,|\,x = r\cos\varphi, y = r\sin\varphi\,, (r,z) \in \Gamma_i, \varphi \in \langle 0, 2\pi)\}, \qquad i = 1,2.$$

The Navier-Stokes equations in the cylindrical coordinate system (r,φ,z) rotating about the z-axis with angular velocity Ω_0 are (see e.g. [1])

$$\begin{aligned}&\frac{\partial w_r}{\partial t} + w_r\frac{\partial w_r}{\partial r} + \frac{w_\varphi}{r}\frac{\partial w_r}{\partial \varphi} + w_z\frac{\partial w_r}{\partial z} - \frac{w_\varphi^2}{r} - 2\Omega_0 w_\varphi - \Omega_0^2 r = \\ &\quad = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\,(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\varphi\varphi}}{r}\right],\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\frac{\partial w_\varphi}{\partial t} + w_r\frac{\partial w_\varphi}{\partial r} + \frac{w_\varphi}{r}\frac{\partial w_\varphi}{\partial \varphi} + w_z\frac{\partial w_\varphi}{\partial z} + \frac{w_r w_\varphi}{r} + 2\Omega_0 w_r = \\ &\quad = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\,(r\tau_{\varphi r})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{\tau_{\varphi r}}{r}\right],\end{aligned} \tag{1.2}$$

$$\frac{\partial w_z}{\partial t} + w_r \frac{\partial w_z}{\partial r} + \frac{w_\varphi}{r}\frac{\partial w_z}{\partial \varphi} + w_z \frac{\partial w_z}{\partial z} = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\left(r\tau_{zr}\right)}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\varphi}}{\partial \varphi} + \frac{\partial \tau_{zz}}{\partial z}\right] . \tag{1.3}$$

Here w_r, w_φ and w_z are radial, circumferential and axial velocities, p is pressure, ϱ is density and τ_{rr}, $\tau_{r\varphi}$, τ_{rz}, $\tau_{\varphi r}$, $\tau_{\varphi\varphi}$, $\tau_{\varphi z}$, τ_{zr}, $\tau_{z\varphi}$ and τ_{zz} are stress tensor components,

$$\begin{aligned}
&\tau_{rr} = -p + 2\mu\frac{\partial w_r}{\partial r}, && \tau_{r\varphi} = \tau_{\varphi r} = \mu\left(\frac{1}{r}\frac{\partial w_r}{\partial \varphi} + \frac{\partial w_\varphi}{\partial r} - \frac{1}{r}w_\varphi\right), \\
&\tau_{\varphi\varphi} = -p + 2\mu\left(\frac{1}{r}\frac{\partial w_\varphi}{\partial \varphi} + \frac{1}{r}w_r\right), && \tau_{\varphi z} = \tau_{z\varphi} = \mu\left(\frac{\partial w_\varphi}{\partial z} + \frac{1}{r}\frac{\partial w_z}{\partial \varphi}\right), \\
&\tau_{zz} = -p + 2\mu\frac{\partial w_z}{\partial z}, && \tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right),
\end{aligned} \tag{1.4}$$

μ is dynamic viscosity.

Moreover, the velocities fulfill the continuity equation

$$\frac{1}{r}\frac{\partial\left(r w_r\right)}{\partial r} + \frac{1}{r}\frac{\partial w_\varphi}{\partial \varphi} + \frac{\partial w_z}{\partial z} = 0 . \tag{1.5}$$

The Navier-Stokes equations and the continuity equation hold in the domain V. On the boundary S_1 we prescribe velocities (essential boundary conditions)

$$w_r = \bar{w}_r , \quad w_\varphi = \bar{w}_\varphi , \quad w_z = \bar{w}_z \tag{1.6}$$

and on the boundary S_2 we prescribe the stress vector components (natural boundary conditions)

$$\tau_{rr}n_r + \tau_{rz}n_z = \bar{p}_r , \quad \tau_{\varphi r}n_r + \tau_{\varphi z}n_z = \bar{p}_\varphi , \quad \tau_{zr}n_r + \tau_{zz}n_z = \bar{p}_r . \tag{1.7}$$

Here $\mathbf{n} = (n_r, n_\varphi, n_z)$, $n_\varphi = 0$, is the unit outward normal vector to the boundary S_2. Bars in (1.6) and (1.7) denote known data.

2. The linear stability

Let us suppose that the steady base flow, whose stability we examine, is axially symmetric, described by functions

$$w_{0r}(r,z) , \quad w_{0\varphi}(r,z) , \quad w_{0z}(r,z) , \quad p_0(r,z).$$

To investigate the stability of the base flow to disturbances, equations that govern the evolution of these perturbations are required. To this end, the base flow is perturbed by a disturbance velocity and pressure, i.e.

$$(w_r, w_\varphi, w_z, p) = (w_{0r}, w_{0\varphi}, w_{0z}, p_0) + \varepsilon(v_r, v_\varphi, v_z, \sigma) , \tag{2.1}$$

and we examine, whether $(w_r, w_\varphi, w_z, p) \to (w_{0r}, w_{0\varphi}, w_{0z}, p_0)$ for $t \to \infty$. Disturbances v_r, v_φ, v_z and σ depend on time t and on all three space variables r, φ and z, i.e.

$$v_r = v_r(t, r, \varphi, z)\,, \quad v_\varphi = v_\varphi(t, r, \varphi, z)\,, \quad v_z = v_z(t, r, \varphi, z)\,, \quad \sigma = \sigma(t, r, \varphi, z)\,.$$

If we substitute from (2.1) into equations (1.1)-(1.5), take into account the fact that the stationary flow functions w_{0r}, $w_{0\varphi}$, w_{0z}, p_0 satisfy those equations and if we neglect terms containing ε^2, we obtain linearized Navier-Stokes equations

$$\frac{\partial v_r}{\partial t} + w_{0r}\frac{\partial v_r}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_r}{\partial \varphi} + w_{0z}\frac{\partial v_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}v_r + \frac{\partial w_{0r}}{\partial z}v_z - \frac{2}{r}w_{0\varphi}v_\varphi - 2\Omega_0 v_\varphi =$$

$$= \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\,(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\varphi\varphi}}{r}\right], \qquad (2.2)$$

$$\frac{\partial v_\varphi}{\partial t} + w_{0r}\frac{\partial v_\varphi}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_\varphi}{\partial \varphi} + w_{0z}\frac{\partial v_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}v_r + \frac{\partial w_{0\varphi}}{\partial z}v_z + \frac{w_{0r}}{r}v_\varphi + \frac{w_{0\varphi}}{r}v_r + 2\Omega_0 v_r =$$

$$= \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\,(r\tau_{\varphi r})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{\tau_{\varphi r}}{r}\right], \qquad (2.3)$$

$$\frac{\partial v_z}{\partial t} + w_{0r}\frac{\partial v_z}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_z}{\partial \varphi} + w_{0z}\frac{\partial v_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}v_r + \frac{\partial w_{0z}}{\partial z}v_z =$$

$$= \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial\,(r\tau_{zr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\varphi}}{\partial \varphi} + \frac{\partial \tau_{zz}}{\partial z}\right]. \qquad (2.4)$$

Stresses in (2.2)-(2.4) have the form (1.4), where we replace p by σ and w by v. The continuity equation reads

$$\frac{1}{r}\frac{\partial\,(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0\,. \qquad (2.5)$$

Velocities v_r, v_φ, v_z and pressure σ fulfill homogeneous boundary conditions

$$v_r = v_\varphi = v_z = 0 \quad \text{on } S_1, \qquad (2.6)$$

$$\tau_{rr}n_r + \tau_{rz}n_z = 0\,, \quad \tau_{\varphi r}n_r + \tau_{\varphi z}n_z = 0\,, \quad \tau_{zr}n_r + \tau_{zz}n_z = 0 \quad \text{on } S_2. \qquad (2.7)$$

The stability problem, therefore, consists in verifying whether

$$(v_r, v_\varphi, v_z, \sigma) \to (0, 0, 0, 0) \qquad \text{pro } t \to \infty. \qquad (2.8)$$

3. The eigenvalue problem

Let $I = \sqrt{-1}$ is the imaginary unit and n is a natural number. Using the transformation

$$(v_r, v_\varphi, v_z, \sigma) = e^{\lambda t + In\varphi}(u_r, u_\varphi, u_z, h)\,, \qquad (3.1)$$

where u_r, u_φ, u_z and h are functions of the two variables r, z and λ is a complex number, we separate out the time t as well as the azimuthal coordinate φ. For $n = 0$ we obtain the rotationally symmetric flow. We insert (3.1) into (2.2)-(2.5), remove the term $e^{\lambda t + In\varphi}$ and obtain the eigenvalue problem in the (r, z)-plane (λ is an eigenvalue and $\mathbf{u} := (u_r, u_\varphi, u_z, h)$ is an eigenfunction)

$$\begin{aligned}\lambda u_r + w_{0r}\frac{\partial u_r}{\partial r} + \frac{nI}{r}w_{0\varphi}u_r + w_{0z}\frac{\partial u_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}u_r + \frac{\partial w_{0r}}{\partial z}u_z - \\ - \frac{2}{r}w_{0\varphi}u_\varphi - 2\Omega_0 u_\varphi = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{nI}{r}\tau_{r\varphi} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\varphi\varphi}}{r}\right],\end{aligned} \tag{3.2}$$

$$\begin{aligned}\lambda u_\varphi + w_{0r}\frac{\partial u_\varphi}{\partial r} + \frac{nI}{r}w_{0\varphi}u_\varphi + w_{0z}\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}u_r + \frac{\partial w_{0\varphi}}{\partial z}u_z + \\ + \frac{w_{0r}}{r}u_\varphi + \frac{w_{0\varphi}}{r}u_r + 2\Omega_0 u_r = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial (r\tau_{\varphi r})}{\partial r} + \frac{nI}{r}\tau_{\varphi\varphi} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{\tau_{\varphi r}}{r}\right],\end{aligned} \tag{3.3}$$

$$\begin{aligned}\lambda u_z + w_{0r}\frac{\partial u_z}{\partial r} + \frac{nI}{r}w_{0\varphi}u_z + w_{0z}\frac{\partial u_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}u_r + \frac{\partial w_{0z}}{\partial z}u_z = \\ = \frac{1}{\varrho}\left[\frac{1}{r}\frac{\partial (r\tau_{zr})}{\partial r} + \frac{nI}{r}\tau_{z\varphi} + \frac{\partial \tau_{zz}}{\partial z}\right],\end{aligned} \tag{3.4}$$

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0\,. \tag{3.5}$$

The stress components now are

$$\begin{aligned}\tau_{rr} &= -h + 2\mu\frac{\partial u_r}{\partial r}, & \tau_{r\varphi} = \tau_{\varphi r} &= \mu\left(\frac{nI}{r}u_r + \frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi\right), \\ \tau_{\varphi\varphi} &= -h + 2\mu\left(\frac{nI}{r}u_\varphi + \frac{1}{r}u_r\right), & \tau_{\varphi z} = \tau_{z\varphi} &= \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{nI}{r}u_z\right), \\ \tau_{zz} &= -h + 2\mu\frac{\partial u_z}{\partial z}, & \tau_{zr} = \tau_{rz} &= \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right).\end{aligned} \tag{3.6}$$

Further, we derive the variational formulation. Let ψ_r, ψ_φ, ψ_z and q are test functions of the two variables r and z and let

$$\psi_r = \psi_\varphi = \psi_z = 0 \qquad \text{on } \Gamma_1. \tag{3.7}$$

We multiply the equation (3.2) by $r\psi_r$, the equation (3.3) by $r\psi_\varphi$, the equation (3.4) by $r\psi_z$, the equation (3.5) by rq, integrate over D and apply the Green theorem (with respect to terms containing stresses τ_{rr}, τ_{rz}, $\tau_{\varphi r}$, $\tau_{\varphi z}$, τ_{zr}, τ_{zz}). If we take into account (2.7) and (3.7), we obtain

$$\int_D \left\{ \left[\lambda u_r + w_{0r}\frac{\partial u_r}{\partial r} + \frac{nI}{r}w_{0\varphi}u_r + w_{0z}\frac{\partial u_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}u_r + \frac{\partial w_{0r}}{\partial z}u_z - \frac{2}{r}w_{0\varphi}u_\varphi - \right. \right.$$
$$\left. \left. - 2\Omega_0 u_\varphi \right] \psi_r + \frac{1}{\varrho}\left[\tau_{rr}\frac{\partial \psi_r}{\partial r} - \frac{nI}{r}\tau_{r\varphi}\psi_r + \tau_{rz}\frac{\partial \psi_r}{\partial z} + \frac{\tau_{\varphi\varphi}}{r}\psi_r \right] \right\} r\,\mathrm{d}r\,\mathrm{d}z = 0\,, \tag{3.8}$$

$$\int_D \left\{ \left[\lambda u_\varphi + w_{0r}\frac{\partial u_\varphi}{\partial r} + \frac{nI}{r}w_{0\varphi}u_\varphi + w_{0z}\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}u_r + \frac{\partial w_{0\varphi}}{\partial z}u_z + \frac{w_{0r}}{r}u_\varphi + \right. \right. \tag{3.9}$$
$$\left. \left. + \frac{w_{0\varphi}}{r}u_r + 2\Omega_0 u_r \right] \psi_\varphi + \frac{1}{\varrho}\left[\tau_{\varphi r}\frac{\partial \psi_\varphi}{\partial r} - \frac{nI}{r}\tau_{\varphi\varphi}\psi_\varphi + \tau_{\varphi z}\frac{\partial \psi_\varphi}{\partial z} - \frac{\tau_{r\varphi}}{r}\psi_\varphi \right] \right\} r\,\mathrm{d}r\,\mathrm{d}z = 0\,,$$

$$\int_D \left\{ \left[\lambda u_z + w_{0r}\frac{\partial u_z}{\partial r} + \frac{nI}{r}w_{0\varphi}u_z + w_{0z}\frac{\partial u_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}u_r + \frac{\partial w_{0z}}{\partial z}u_z \right] \psi_z + \right.$$
$$\left. + \frac{1}{\varrho}\left[\tau_{zr}\frac{\partial \psi_z}{\partial r} - \frac{nI}{r}\tau_{z\varphi}\psi_z + \tau_{zz}\frac{\partial \psi_z}{\partial z} \right] \right\} r\,\mathrm{d}r\,\mathrm{d}z = 0\,, \tag{3.10}$$

$$\int_D \left\{ \frac{\partial u_r}{\partial r} + \frac{nI}{r}u_\varphi + \frac{\partial u_z}{\partial z} + \frac{u_r}{r}q \right\} r\,\mathrm{d}r\,\mathrm{d}z = 0\,. \tag{3.11}$$

Moreover, we add essential boundary conditions

$$u_r = u_\varphi = u_z = 0 \qquad \text{on } \Gamma_1. \tag{3.12}$$

The stability expressed by the relation (2.8) occurs if all eigenvalues λ of the problem (3.8) - (3.12) have negative real parts.

4. The discretization

The approximate finite-dimensional eigenvalue problem is obtained by the spectral element method. The algorithm was implemented in MATLAB. The generalized eigenvalue problem

$$\mathbf{Au} = \lambda \mathbf{Bu} \tag{4.1}$$

was solved by means of MATLAB function `eig`. In what follows we describe the intimated process more detaily.

The spectral element method is a discretization technique, which has excellently approved as a solution method for incompressible flows problems, see e.g. [2], [3], [8], [6]. We decided to use this method mainly because the spectral element approximation of the eigenvalue problem promise to be sufficiently representative already at a low order of matrices $\mathbf{A}$ and $\mathbf{B}$, as theory as well as a practise indicate. Let us show in brief how an approximate eigenvalue problem can be obtained.

We restrict ourselves to the case that D is a rectangle,

$$D = \{(r, z) \,|\, 0 < R_1 < r < R_2,\ 0 < z < L\}\,. \tag{4.2}$$

We divide D into $n_r \times n_z$ concurrent rectangular elements D^{ij}, $i = 1, 2, \dots, n_r$, $j = 1, 2, \dots, n_z$, with sides of length $d_r = (R_2 - R_1)/n_r$ and $d_z = L/n_z$ in the direction of the r-axis and z-axis, respectively. The corresponding partition of the interval $\langle R_1, R_2\rangle$ and $\langle 0, L\rangle$ is determined by points

$$r_i = R_1 + i d_r\,,\ i = 0, 1, \dots, n_r\,, \qquad z_j = j d_z\,,\ j = 0, 1, \dots, n_z\,.$$

The element D^{ij} is the image of the reference element $\hat{D} = (-1, 1)^2$ under the transformation

$$r^{ij}(\xi) = R_1 + \left[i - 1 + \tfrac{1}{2}(\xi + 1)\right] d_r\,,\ z^{ij}(\eta) = \left[j - 1 + \tfrac{1}{2}(\eta + 1)\right] d_z\,, \qquad [\xi, \eta] \in \hat{D}\,. \tag{4.3}$$

If

$$-1 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$$

are nodes of the Gauss-Legendre-Lobatto quadrature formula (briefly GLL formula) of order $2N - 1$, see e.g. [2], then images $[r_k^{ij}, z_\ell^{ij}]$ of nodes $[\xi_k, \xi_\ell]$ under the transformation (4.3),

$$r_k^{ij} = r^{ij}(\xi_k)\,, \qquad z_\ell^{ij} = z^{ij}(\xi_\ell)\,, \qquad k, \ell = 0, 1, \dots, N,$$

are nodes of the product GLL quadrature formula on the element D^{ij}.

We approximate velocities u_r, u_φ and u_z with piecewise polynomial functions $u_{r,\vartheta}$, $u_{\varphi,\vartheta}$ and $u_{z,\vartheta}$, which are on every element D^{ij} interpolation polynomials $u^{ij}_{r,\vartheta}$, $u^{ij}_{\varphi,\vartheta}$ and $u^{ij}_{z,\vartheta}$ of degree N uniquely determined by their values $u^{ij}_{r,k\ell}$, $u^{ij}_{\varphi,k\ell}$, $u^{ij}_{z,k\ell}$ at the nodes $[r_k^{ij}, z_\ell^{ij},]_{k,\ell=0}^N$. The lower index ϑ specifies the quality of the approximation: $\vartheta = (d, N)$, where $d = \max(d_r, d_z)$ represents the fineness of partition and N is the degree of polynomial approximation.

Similarly, we introduce nodes $[\tilde{r}_k^{ij}, \tilde{z}_\ell^{ij}]$ of the product Gauss-Legendre quadrature formula (briefly GL formula) on the element D^{ij},

$$\tilde{r}_k^{ij} = r^{ij}(\tilde{\xi}_k)\,, \qquad \tilde{z}_\ell^{ij} = z^{ij}(\tilde{\xi}_\ell)\,, \qquad k, \ell = 1, \dots, N - 1,$$

where

$$-1 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_{N-2} < \tilde{\xi}_{N-1} < 1$$

are nodes of the GL formula of order $2N-3$, see e.g. [2]. We approximate pressure h with piecewise polynomial function $h_{\tilde{\vartheta}}$, which is on every element D^{ij} an interpolation polynomial $h^{ij}_{\tilde{\vartheta}}$ of degree $N - 2$ uniquely determined by its values $\tilde{h}^{ij}_{k\ell}$ at nodes $[\tilde{r}_k^{ij}, \tilde{z}_\ell^{ij},]_{k,\ell=1}^{N-1}$, where $\tilde{\vartheta} = (d, N - 2)$.

The fulfillment of boundary conditions can be achieved by zeroing interpolants at nodes lying on $\bar{\Gamma}_1$.

Further, we replace stationary velocities w_{0r}, $w_{0\varphi}$ and w_{0z} by their interpolations $u_{0r,\vartheta}$, $u_{0\varphi,\vartheta}$ and $u_{0z,\vartheta}$, which are again piecewise polynomials of degree N, whose values $w^{ij}_{0r,k\ell}$, $w^{ij}_{0\varphi,k\ell}$ and $w^{ij}_{0z,k\ell}$ at nodes $[r^{ij}_k, z^{ij}_\ell]$ are supposed to be known.

We also replace test functions ψ_r, ψ_φ, ψ_z by their interpolants $\psi_{r,\vartheta}$, $\psi_{\varphi,\vartheta}$, $\psi_{z,\vartheta}$ (which are piecewise polynomials of degree N, on D^{ij} determined by their values $\psi^{ij}_{r,k\ell}$, $\psi^{ij}_{\varphi,k\ell}$, $\psi^{ij}_{z,k\ell}$ at nodes $[r^{ij}_k, z^{ij}_\ell]$) and q by the interpolant q_ϑ (which is a piecewise polynomial of degree $N-2$, on D^{ij} determined by its values $\tilde{q}^{ij}_{k\ell}$ at nodes $[\tilde{r}^{ij}_k, \tilde{z}^{ij}_\ell]$). Test functions $\psi_{r,\vartheta}$, $\psi_{\varphi,\vartheta}$, $\psi_{z,\vartheta}$ equal to zero in GLL nodes lying on the boundary $\bar{\Gamma}_1$.

We insert all interpolants into equations (3.8)-(3.11) and (3.6), express integrals over the whole domain D as a sum of integrals over individual elements D^{ij} and every from those integrals compute numerically: integrals containing pressure $h_{\tilde{\vartheta}}$ or test function $q_{\tilde{\vartheta}}$ with the GL formula and remaining integrals with the GLL formula.

We rearrange the matrix $\{u^{ij}_{r,k\ell}\}^N_{k,\ell=0}$ of parameters determining the function $u^{ij}_{r,\vartheta}$ into the column vector

$$\mathbf{u}^{ij}_r = (u^{ij}_{r,00}, u^{ij}_{r,10}, \ldots, u^{ij}_{r,N0}, u^{ij}_{r,01}, u^{ij}_{r,11}, \ldots, u^{ij}_{r,N1}, \ldots, u^{ij}_{r,0N}, u^{ij}_{r,1N}, \ldots, u^{ij}_{r,NN})^T.$$

Vectors of parameters $\mathbf{u}^{ij}_\varphi$, $\mathbf{u}^{ij}_z$ and $\tilde{\mathbf{h}}^{ij}$ will be obtained similarly. We merge all unknown parameters into one column vector $\mathbf{u}^{ij}$,

$$[\mathbf{u}^{ij}]^T = ([\mathbf{u}^{ij}_r]^T, [\mathbf{u}^{ij}_\varphi]^T, [\mathbf{u}^{ij}_z]^T, [\tilde{\mathbf{h}}^{ij}]^T)^T.$$

If

$$[\boldsymbol{\psi}^{ij}]^T = ([\boldsymbol{\psi}^{ij}_r]^T, [\boldsymbol{\psi}^{ij}_\varphi]^T, [\boldsymbol{\psi}^{ij}_z]^T, [\mathbf{q}^{ij}]^T)^T$$

is the corresponding vector of test functions parameters, then summing all approximate equations we arrive at the equation

$$\sum_{i=1}^{n_r} \sum_{j=1}^{n_z} [\boldsymbol{\psi}^{ij}]^T (\lambda \mathbf{B}^{ij} - \mathbf{A}^{ij}) \mathbf{u}^{ij} = 0.$$

Arranging all nonzero components of vectors $\mathbf{u}^{ij}$ and $\boldsymbol{\psi}^{ij}$ into column vectors $\mathbf{u}$ and $\boldsymbol{\psi}$ enables to write the preceding equation in the form

$$\boldsymbol{\psi}^T (\lambda \mathbf{B} - \mathbf{A}) \mathbf{u} = 0\,,$$

and as the vector $\boldsymbol{\psi}$ is arbitrary, we obtain the generalized eigenvalue problem (4.1).

The matrix $\mathbf{A}$ is regular, nonhermitian, real for $n = 0$ and complex for $n > 0$. The matrix $\mathbf{B}$ is diagonal and singular: diagonal coefficients corresponding to parameters $\mathbf{q}^{ij}$ are equal to zero, remaining diagonal coefficients are real and positive.

5. Numerical examples

We choose the data having regard to the following technical problem. Water enter the pipe through the boundary $\Gamma_{in} = \bar{D} \cap \{z = 0\}$ and leaves it through the boundary $\Gamma_{out} = \bar{D} \cap \{z = L\}$. $\Gamma_{wall} = \bar{D} \cap \{r = R_2\}$ is the wall of the pipe and $\Gamma_{rope} = \bar{D} \cap \{r = R_1\}$ is the interface between water and the cavitating vapour rope. In this case we have

$$\begin{aligned} \Gamma_1 &= \Gamma_{in} \cup \Gamma_{wall} = \{(r,z) \in \Gamma \,|\, z = 0 \text{ or } r = R_2\}\,, \\ \Gamma_2 &= \Gamma_{out} \cup \Gamma_{rope} = \{(r,z) \in \Gamma \,|\, z = L \text{ or } r = R_1\}\,. \end{aligned} \tag{5.1}$$

Further, we chose $R_1 = 0.015$, $R_2 = 0.15$, $L = 0.5$ and $\varrho = 10^3$. Dynamical viscosity of water $\mu = 10^{-3}$, but we will experiment also with another values of μ.

Example 1. We consider a hypothetical flow as a solid body movement. We take $w_{0r} = 0$, $w_{0\varphi} = 0$, $w_{0z} = C_0$, use the boundary partitition (5.1) and experiment with values of $C_0 \geq 0$, Ω_0, μ, n, n_r, n_z and N. The flow of this type is stable and numerical computations confirm this. With $\mu = 0$ we have the inviscid flow, corresponding numerical results were published in [5] (for $\mu = 0$ we have only the condition $u_r = 0$ on Γ_{wall}). When μ increases, then $\max \mathrm{Re}(\lambda)$ increases as well, i.e. the stability grows. In the table 1 we introduce typical results.

μ	10^{-3}	10^{-2}	10^{-1}	1	10^1	10^2	10^3
$\max \mathrm{Re}(\lambda)$	-0.30	-0.36	-0.84	-3.78	-11.30	-70.10	-665.26

Table 1: The stability increase, $n = 0$, $C_0 = \Omega_0 = 1$, $n_r = n_z = 1$, $N = 8$.

Example 2. We consider the flow between two concentric rotating cylinders. By Ω_1 and Ω_2 we denote the angular velocity of the inner and outer cylinder, respectively. Then

$$w_{0r} = w_{0z} = 0\,, \qquad w_{0\varphi} = \frac{(R_2^2\Omega_2 - R_1^2\Omega_1)r^2 + R_1^2R_2^2(\Omega_1 - \Omega_2)}{(R_2^2 - R_1^2)r}\,, \tag{5.2}$$

see e.g.[7]. The stability of this flow is known as the Taylor-Couette problem.

(a) If we consider an inviscid flow, i.e. $\mu = 0$, then the classical Taylor's result claims: the flow is stable if and only if

$$\frac{\Omega_2}{\Omega_1} > \frac{R_1^2}{R_2^2}\,, \tag{5.3}$$

see e.g. [7]. Numerical computations confirm this theoretical result. We take

$$\Gamma_1 = \bar{D} \cap (\{z = 0\} \cup \{r = R_1\} \cup \{r = R_2\})\,, \qquad \Gamma_2 = \bar{D} \cap \{z = L\} \tag{5.4}$$

and set $\Omega_1 = 1$. Then the stability should arise for

$$\Omega_2 > \frac{0.015^2}{0.15^2} = 0.01.$$

We take $n = 0$, $N = 4$, $n_z = 1$ and find that the lowest value of Ω_2, for which the program gives nonnegative real parts of all eigenvalues, converge to 0.01 if n_r increases, see Table 2 (values of $\min \Omega_2$ are rounded to 4 decimal places).

n_r	1	2	4	8	16	32
$\min \Omega_2$	0.0418	0.0328	0.0200	0.0126	0.0104	0.0100

Table 2: The lowest attainable angular velocity Ω_2, $n_z = 1$, $N = 4$.

(b) In case of viscous flows the stability significantly increases. We carried out the following numerical experiment: for $\mu = 1$, $n = 0$, $n_r = n_z = 1$, $N = 8$ and the boundary partition (5.4) we took several values of Ω_2 and for each of them we found the critical value $\Omega_{1,crit}(\Omega_2)$, where the onset of instability occurs (i.e. $\Omega_{1,crit}(\Omega_2)$ is the point of transition from the stable to unstable flow regime for a given Ω_2). Results in the (Ω_2, Ω_1)-phase plane are presented in Figure 1. We have found that for $\Omega_1 < 43$ the Couette flow remains stable for all values of Ω_2. The dotted line denotes the stability border for $\mu = 0$. Our numerical results qualitatively agree with well known Taylor's experimental results, see e.g. [7].

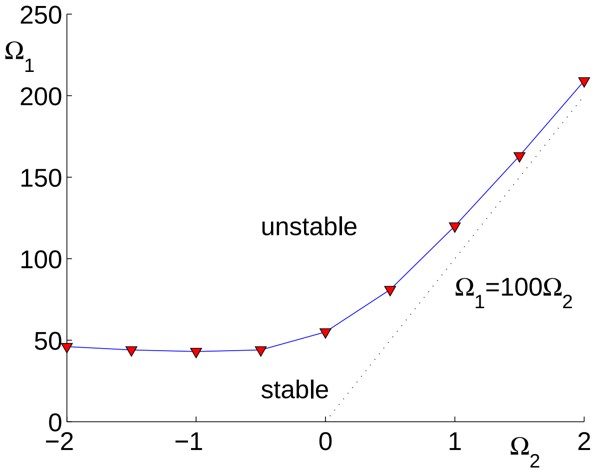

Figure 1: The stability of Couette flow.

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References

[1] M. Brdička, L. Samek, B. Sopko: *Mechanika kontinua,* Academia, Praha, 2000.

[2] M. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang: *Spectral Methods. Fundamentals in Single Domains,* Springer, Berlin, 2006.

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang: *Spectral Methods. Evolution to Complex Geometries and Applications to Fluid Dynamics,* Springer, Berlin, 2007.

[4] W. O. Criminale, T. L. Jackson, R. D. Joslin: *Theory and Computation in Hydrodynamic Stability,* Cambridge University Press, Cambdridge, 2003.

[5] L. Čermák: *Stabilita stacionárního proudění v rotačně symetrické válcové oblasti,* [online], 6. Matematický workshop, FAST VUT Brno, [cit. 10.08.2008]. Available from www: <http://math.fce.vutbr.cz/konference>.

[6] M. O. Deville, P. F. Fisher, E. H. Mund: *High-Order Methods for Incompressible Fluid Flow,* Cambridge University Press, Cambridge, 2002.

[7] P. G. Drazin, W. H. Reid: *Hydrodynamic stability,* Cambridge University Press, Cambdridge, 2004.

[8] G. E. Karniadakis, S. J. Sherwin: *Spectral/hp Element Methods for Computational Fluid Dynamics,* Oxford University Press, Oxford, 2005.

[9] V. Theofilis: *Advances in global linear instability analysis of nonparalel and three-dimensional flows,* Prog. Aero. Sci 39 (2003), pp. 249-315.