Positive and Oscillating Solutions of Linear Discrete Equations

Jaromír Baštinec
Faculty of Electrical Engineering and Communication, Brno University of Technology, Czech Republic
e-mail: bastinec@feec.vutbr.cz

Josef Diblík
Brno University of Technology, Czech Republic
e-mail: diblik.j@fce.vutbr.cz, diblik@feec.vutbr.cz

We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$ where $s = -\infty$ and $q = \infty$ are admitted, too. The topic of our study is a linear scalar discrete equation of k-th order

$$\Delta x(n) = -p(n)x(n-k), \tag{1}$$

where $p\colon \mathbb{Z}_a^\infty \to (0,\infty)$, $k \ge 1$, a is an integer and $n \in \mathbb{Z}_a^\infty$. Let $\varphi\colon \mathbb{Z}_{a-k}^a \to \mathbb{R}$. Together with discrete equation (1), we consider an initial problem: determine a solution $x = x(n)$ of equation (1) satisfying the initial conditions

$$x(n) = \varphi(n),\ n \in \mathbb{Z}_{a-k}^a \tag{2}$$

with prescribed real constants $\varphi(n)$.

A solution of initial problem (1), (2) is defined as an infinite sequence of numbers $\{x^n\}_{n=-k}^\infty$ with $x^n = x(a+n)$, i.e., $\{x^{-k} = \varphi(a-k), \dots, x^0 = \varphi(a), x^1 = x(a+1), \dots, x^n = x(a+n), \dots\}$ such that, for any $n \in \mathbb{Z}_a^\infty$, equality (1) holds.

Solution of (1), (2) is called *positive* if $x(n) > 0$ for every $n \in \mathbb{Z}_{a-k}^\infty$. Solution of initial problem (1), (2) is called *oscillating* on $\mathbb{Z}_{a-k}^\infty$ if for arbitrary $m \in \mathbb{Z}_{a-k}^\infty$ there exists $n \ge m$ such that $x(n)x(n+1) \le 0$.

We define auxiliary functions p, ν as

$$P(n) := \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2}\right] \tag{3}$$

and

$$P_\theta(n) := \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k\theta}{8(n\ln n)^2}\right] \tag{4}$$

which play an important role in the investigation of positive and oscillating solutions of equation (1). We assume that n in (3) and (4) is sufficiently large for $P(n)$ and $P_\theta(n)$ to be well defined.

Theorem 1 *If*

$$p(n) \le P(n)$$

for all $n \in \mathbb{Z}_m^\infty$ where m is sufficiently large, then exists a positive solution $x = x(n)$, $n \in \mathbb{Z}_m^\infty$ of equation (1).

Theorem 2 *If there exists a $\theta \in (1,\infty)$ such that*

$$p(n) > P_\theta(n) \tag{5}$$

for all $n \in \mathbb{Z}_m^\infty$ where m is sufficiently large, then all solutions of (1) are oscillating on $\mathbb{Z}_m^\infty$.

The proof of Theorem 1 is based on a method developed in [2, 3, 5, 6]. The proof of Theorem 2 uses a method suggested in [9].

Acknowledgement. This research was partially supported by the grant 201/07/0145 of Czech Grant Agency (Prague), Council of Czech Government MSM 00216 30503, MSM 0021630519 and MSM 00216 30529.

References

[1] R.P. Agarwal, M. Bohner, Wan-Tong Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations,* Marcel Dekker, Inc., 2004.

[2] J. Baštinec, J. Diblík, Remark on positive solutions of discrete equation $\Delta u(k+n) = -p(k)u(k)$, *Nonlinear Anal.*, **63** (2005), e2145–e2151.

[3] J. Baštinec, J. Diblík, B. Zhang, Existence of bounded solutions of discrete delayed equations, *Proceedings of the Sixth International Conference on Difference Equations*, CRC, Boca Raton, FL, 359–366, 2004.

[4] L. Berezansky, E. Braverman, On existence of positive solutions for linear difference equations with several delays, *Adv. Dyn. Syst. Appl.*, **1** (2006), 29–47.

[5] J. Diblík, Anti-Lyapunov method for systems of discrete equations, *Nonlinear Anal.,* **57** (2004), 1043–1057.

[6] J. Diblík, Asymptotic behavior of solutions of discrete equations, *Funct. Differ. Equ.*, **11** (2004), 37–48.

[7] J. Diblík, Positive and oscillating solutions of differential equations with delay in critical case, *J. Comput. Appl. Math.* **88** (1998), 185–202.

[8] J. Diblík, I. Růžičková, M. Růžičková, A general version of the retract method for discrete equations, *Acta Math. Sin.*, **23** (2007), No 2, 341–348.

[9] Y. Domshlak, Delay-difference equations with periodic coefficients: Sharp results in oscillation theory, *Math. Inequal. Appl.*, **1** (1998), 1331–1343.

[10] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations,* Clarendon Press (1991).