Asymptotic properties of one differential equation with unbounded delay

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Abstract

This contribution deals with asymptotic behavior of solutions of a differential equation with unbounded delay. This equation includes as the special cases some equations which have been recently considered , for example the logistic equation with recruitment delays, which was considered by Gopalsamy[2]. The purpose of this paper is the existence a solution of this equation which may be at $t \to \infty$ represented by asymptotic series. To prove this results is used the modification of the Ważewski's topological method.
The first Lyapunoff's method is often used to construct the solution of ordinary differential equations in the form of power–like series. Such a way is not possible here. First lefthand ends of existence intervals of partial sums tend to infinity and, secondly, if ti does not happen, the partial sums need not converge uniformly.

1 Introduction

In this contribution we consider the asymptotic properties of solutions of delayed differential equation

$$g(t)\dot{y}(t) = -ay(t) + \sum_{|\mathbf{i}|=2}^{N} c_{\mathbf{i}}(t)\big(\mathbf{y}(\boldsymbol{\xi}(t))\big)^{\mathbf{i}}, \tag{1}$$

where a is a positive constant, $N \geq 2$ is a integer, $\mathbf{i} = (i_1, \dots, i_n)$ is a multiindex, ($i_j \geq 0$ are integers) $|\mathbf{i}| = \sum_{j=1}^{n} i_j$, and $\mathbf{y}(\boldsymbol{\xi}(t))^{\mathbf{i}} = \big(y(\xi_1(t)), \dots, y(\xi_n(t))\big)^{(i_1,\dots,i_n)} = \prod_{j=1}^{n}\big(y(\xi_j(t))\big)^{i_j}$.

The continuous functions $\xi_j(t)$ satisfy $T_0 \leq \xi_j(t) \leq t$ for all $t \in [t_0, \infty)$. Further conditions for the continuous functions $g(t):\ [t_0,\infty) \to R_+$ and $c_{\mathbf{i}}(t):\ [t_0,\infty) \to R$ will be given latter.

The purpose of this paper is study the asymptotical properties of solutions of the equation (1). We find conditions for existence solution $y(t) = y(t, C)$ of (1) which may be at $t \to \infty$ represented by asymptotic series (symbol $\approx$ denotes the asymptotic expansions)

$$y(t) \approx \sum_{k=1}^{\infty} f_k(t)\varphi^k(t, C), \tag{2}$$

where $\varphi(t, C)$ is the solution of the homogenous equation $g(t)\dot{y}(t) = -ay(t)$, given by the formula $\varphi(t, C) = C \exp \int\limits_0^t \frac{-a}{g(u)}\, du.$, $f_1(t) \equiv 1$ and the functions $f_k(t)$ for $k = 2, .., n$ are particular solutions of some system of auxiliary differential equations. To prove our results we will use Ważewski's topological method in the form, which is used in [1] for differential equations with unbounded delay and finite memory, for more details about this type of equations see [3].

2 Basic Notion

We define the function $\xi(t) = \min\limits_{1\le i\le n} \xi_i(t)$. Therefore all asymptotic relations will be considered for $t \to \infty$, this fact will be below omitted and also throughout this paper $g(t)$, $G(t)$ denote functions such that

C1. $g(t) \in C^0[0,\infty),\ g(t) > 0$ for $t \ge t_0$ and $g(t) = O(1)$.

C2. $G(t) - G(\xi(t)) = o(G(t))$, where function $G(t)$ is define as $G(t) = \left(\int\limits_0^t g^{-1}(u)\,du\right)^{-1}$

Remark 1 *This conditions enable to consider relative large class of functions. We note that the condition* **C1.** *evokes that the integral* $\displaystyle\int_{t_0}^{\infty} \frac{a\,ds}{g(s)}$ *is divergent and at the first the function* $\varphi(t,C)$ *satisfies* $\varphi^k(t,C) = o\left(\varphi^l(t,C)\right)$ *for* $k > l$ *at the second* $G(t) = o(1)$.
Therefore $G(t) > 0$ *the second condition* **C2.** *is another form of relation* $\lim\limits_{t\to\infty} \dfrac{G(\xi(t))}{G(t)} = 1$. *If moreover yields* $t - \xi(t) \ge r > 0$ *we have* $G(t) = o(g(t))$.

We consider the equation

$$\dot y(t) = y(t) f_1(t) + f_2(t), \tag{3}$$

where $f_1(t) \in C^0[t_0,\infty),\ f_1(t) > 0,\ \left(f_1(t)\right)^{-1} = O(1)$ and also exists the continous function $K(t)$ such that $K(s) - K(t) = O\Big(\int\limits_t^s f_1(u)du\Big)$ and moreover the function $f_2(t)$ fulfils :

1. $\lim\limits_{t\to\infty} f_2(t) \exp\left(-\tau K(t)\right) = 0$ for all positive τ.
2. $\lim\limits_{t\to\infty} f_2(t) \exp\left(\tau K(t)\right) = \infty$ for all positive τ.

Lemma 1 *If the function* $f_2(t)$ *satisfies the assumption* 1 *then there exists at least one solution* $Y(t)$ *of the equation* (3) *satisfying the assumption* 1 *too.*
If the function $f_2(t)$ *more over satisfies the assumption* 2 *then the solution* $Y(t)$ *satisfies assumption* 2 *too.*

Proof :
From the assumptions *1, 2* for sufficient small τ we get

$$A_1 \exp\left(-\tau K(t)\right) \le |Y(t)| \le A_2 \exp\left(\tau K(t)\right),$$

where $Y(t) = \int\limits_t^{\infty} (-f_2(s) \exp \int\limits_t^s -f_1(u)du)ds$ and $A_1,\ A_2$ are suitable constants. The assumption *1* implies the first inequality ensuring the existence of the integral which is the solution of given equation.

For easy specification of coefficients power series which is the product of the power series raised to a power, it is suitable to denote:

$\mathfrak{s} = (\mathfrak{s}_1, \ldots, \mathfrak{s}_n)$ is an ordered n-tuple of these sequences $\mathfrak{s}_j = \left\{\mathfrak{s}_j^k\right\}_{k=1}^{\infty}$, of nonnegative integers with finite sum $|\mathfrak{s}_j| = \sum\limits_{k=1}^{\infty} \mathfrak{s}_j^k$ and
$\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ is an ordered n-tuple of sequences (of numbers or functions) $\mathbf{c}_j = \{c_j^k\}_{k=1}^{\infty}$ and $\mathbf{i}(\mathfrak{s}) = (|\mathfrak{s}_1|, \ldots, |\mathfrak{s}_n|)$

$$V(\mathfrak{s}) = \sum_{j=1}^{n}\sum_{k=1}^{\infty} k\mathfrak{s}_j^k, \qquad \mathfrak{s}! = \prod_{j=1}^{n}\prod_{k=1}^{\infty} \mathfrak{s}_j!^k$$

$$\mathcal{C}^{\mathfrak{s}} = \prod_{j=1}^{n}\prod_{k=1}^{\infty} \left(c_j^k\right)^{\mathfrak{s}_j^k}, \qquad \mathbf{i}(\mathfrak{s})! = \prod_{j=1}^{n} |\mathfrak{s}_i|!,$$

where $\left(c_j^k\right)^0 = 1$ for every c_j^k. Then it is possible to express:

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{\infty} c_j^k x^k\right)^{i_j} = \sum_{k=|\mathbf{i}|}^{\infty} x^k \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}} \frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{C}^{\mathfrak{s}}$$

where symbol $\sum\limits_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}}$ denotes the sum over all $\mathfrak{s}$ such that $V(\mathfrak{s}) = k,\ \mathbf{i}(\mathfrak{s}) = \mathbf{i} = (i_1, \ldots, i_n)$.

3 Main results

Let the formal solution of equation (1) is expressed in the form (2), where $\varphi(t, C)$ is the general solution of the equation $g(t)\dot{y}(t) = -ay(t)$, consequently $\varphi(t, C) = C \exp \int\limits_{t_0}^{t} \frac{-a}{g(s)}\, ds$, where C is a constant and function $f_1(t) = 1,\ f_k(t)$ for $k \geq 2$ are unknown functions for the time being. Substituting $y(t)$ in the equation (1) and comparing coefficients of the same powers $\varphi^k(t, C)$ we obtain an auxiliary system of linear differential equations:

$$g(t)\dot{f}_k(t) = a(k-1)f_k(t) + \sum_{|\mathbf{i}|=2}^{N} c_{\mathbf{i}}(t) \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}} \frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{F}^{\mathfrak{s}}, \text{ where} \qquad (4_k)$$

$$\mathcal{F}(t) = \left(\left\{f_k(\xi_1(t)) \exp \int\limits_{\xi_1(t)}^{t} \frac{ak}{g(u)} du\right\}_{k=1}^{\infty}, \ldots, \left\{f_k(\xi_n(t)) exp \int\limits_{\xi_n(t)}^{t} \frac{ak}{g(u)} du\right\}_{k=1}^{\infty}\right).$$

As $V(\mathfrak{s}) = k \geq 2$ and $|\mathbf{i}(\mathfrak{s})| \geq 2$ we get $\mathfrak{s}_i^l = 0$ for $l \geq k$, therefore the auxiliary system (4_k) is recurrent.

Theorem 1 *Let the functions $c_{\mathbf{i}}(t)$ fulfil*

$$\lim_{t\to\infty} c_{\mathbf{i}}(t) \exp\left\{-\tau \int\limits_{t_0}^{t} \frac{du}{g(u)}\right\} = 0,$$

for all positive τ, then there exists a sequence $\{f_k(t)\}_{k=1}^{\infty}$ of solutions of the auxiliary system (4_k)

$$f^k(t) = \int_t^{\infty} \frac{-a}{g(s)} \sum_{|\mathbf{i}|=2}^{N} c_{\mathbf{i}}(t) \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}} \frac{|\mathbf{i}(\mathfrak{s})|!}{\mathbf{i}(\mathfrak{s})!}\mathcal{F}^{\mathfrak{s}} \exp\left\{-\int\limits_{t}^{s} \frac{a(k-1)}{g(u)} du\right\} ds \qquad (5_k)$$

such that

$$\lim_{t\to\infty} f_k(t) \exp\left\{-\tau \int_{t_0}^{t} \frac{du}{g(u)}\right\} = 0 \textit{ for all positive } \tau.$$

Proof: The formulas (5_k) are obtained by integrating the system (4_k). For the applying lemma 1 we put $K(t) = \int_{t_0}^{t} \frac{a\,du}{g(u)}$. The condition **c 2** gives the fact that for the function $y(t)$ satisfying the assumptions *1* of the lemma 1 the function $y(\xi(t))$ satisfies this assumption too. Therefore the sum and the product of functions verifying the assumptions *1* of the lemma 1 satisfy the assumptions 1 lemma 2, using lemma 2 we can easily show the convergence of (5_k) and desired property.

Remark 2 *Analogous assertion of theorem* 1 *with property which is described by the assumption* 2 *of lemma* 2 *is not possible to prove, for the sum of functions verifying the assumption* 2 *must not satisfies this assumption.*

We shall denote:

$$y_k(t) = \sum_{l=1}^{k} f_l(t)\varphi^l(t, C)$$

Theorem 2 *Let the assumptions of theorem 1 hold and suppose that*

$$\lim_{t\to\infty} f_{k+1}^{-1}(t)\exp(-\tau G^{-1}(t)) = 0$$

where $\tau < 1$ *is a constant.*
Then for every $C \neq 0$ *and* $\psi \in C^0[-r, 0], \|\psi\| \leq 1, \psi(0) = 0$ *there exists a solution* $y_C(t)$ *of equation (1) such that*

$$|y_C(t) - y_k(t)| \leq \delta |f_{k+1}(t)\varphi^{k+1}(t, C)| \tag{7}$$

for $t \in [t_C, \infty)$ *where coefficients* $f_k(t)$ *are the solutions* (5_k) *of the system* (4_k), $\delta > 1$ *is a constant,* t_C *is a function of the parametr* C *and of* δ, k.

Proof: The existence of solution $y_C(t)$ which satisfies the inequality (7) is proved by Ważewski's principle for retarded functional differential equations with unbounded delay with finite memory. For this method see [1]. The prove is analogous with the prove in [4] and the technical details are omitted.

Theorem 3 *Let the assumptions of theorem 1 be satisfied and there exist a sequence* $\{K_k\}_{k=1}^{\infty}$, $K_0 = 1$ *such that assumptions of theorem 2 are satisfied for every* K_k.
then there exists the asymptotic expansion of the solution $y_C(t)$ *in the form*

$$y_C(t) \approx \sum_{k=1}^{\infty} F_k(t)\varphi^{K_k}(t, C),$$

where $F_k(t) = \sum_{l=K_{k-1}}^{K_k - 1} f_l(t)\varphi^{l-K_k}(t, C)$, $f_l(t)$ *are solutions* 5_l *and* $n_0 = 1$ *in the summation of* Y_1.

Proof: As

$$\lim_{t\to\infty} \frac{F_{k+1}(t)\varphi^{k+1}(t, C)}{F_k(t)\varphi^k(t, C)} = \lim_{t\to\infty} \frac{f_{n_{k+1}}(t) + f_{n_{k+1}-1}\varphi(t, C) + \dots}{f_{n_k}(t) + f_{n_k-1}\varphi(t, C) + \dots +}$$
$$\frac{\dots + f_{n_k}(t)\varphi^{n_k}(t, C) f_{n_k}(t)\varphi^{n_{k+1}-n_k}(t, C)}{\dots + f_{n_{k-1}}(t)\varphi^{n_{k-1}-n_k}(t, C))}\varphi^{n_{k+1}-n_k}(t, C) = 0$$

the assertion is proved.

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