Monotone iterative technique for integrodifferential equations

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In recent years the monotone iterative technique coupled with lower and upper solutions has been employed to obtain the existence of extremal solutions (which are limits of monotone sequences) for nonlinear ordinary differential equations, integral and integro-differential equations (see [1-6]). In this paper we determine conditions of the existence of extremal solutions for the following periodic boundary value problem (PBVP)

$$x'(t) = f(t, x(t), Tx(t)), \quad x(0) = x(2\pi), \tag{1}$$

where $f \in C[[0, 2\pi] \times R \times R, R]$, $Tx(t) = \int_0^t K(t, s)x(s)ds$ and $K \in C[[0, 2\pi] \times [0, 2\pi], R^+]$.

Suppose that $\alpha(t)$, $\beta(t)$ are lower and upper solutions of (1) and for convenience we list the following hypotheses that α, β and f must satisfy:

(H_0) $\alpha, \beta \in C^1([0, 2\pi], R)$ such that $\alpha(t) \le \beta(t)$,

$$\alpha' \le f(t, \alpha, T\alpha), \quad \alpha(0) \le \alpha(2\pi)$$

and

$$\beta' \ge f(t, \beta, T\beta), \quad \beta(0) \ge \beta(2\pi);$$

(H_1) for any $v, \overline{v}, \phi, \overline{\phi} \in C([0, 2\pi])$ such that

$$\alpha(t) \le v(t) \le \overline{v}(t) \le \beta(t)$$

and

$$\alpha(t) \le \phi(t) \le \overline{\phi}(t) \le \beta(t)$$

the inequality

$$f(t, v, T\phi) - f(t, \overline{v}, T\overline{\phi}) \ge -M(v - \overline{v}) - NT(\phi - \overline{\phi}),$$

holds for $t \in [0, 2\pi]$ and M, N are positive constants satisfying

$$2Nk_0\pi e^{2M\pi} < M,$$

where $k_0 = \max K(t, s)$ on $[0, 2\pi] \times [0, 2\pi]$.

Theorem 1 *Assume that (H_0, H_1) hold. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that*

$$\lim_{n\to\infty} \alpha_n(t) = g(t) \text{ and } \lim_{n\to\infty} \beta_n(t) = h(t)$$

uniformly on $[0, 2\pi]$, $g(t)$ and $h(t)$ are minimal and maximal solutions of PBVP (1) respectively, satisfying the inequality

$$\alpha(t) \le g(t) \le h(t) \le \beta(t)$$

for $t \in [0, 2\pi]$.

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