Stability of the trivial solution of real two-dimensional differential system with nonconstant delay

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The following special notation will be used throughout the presentation:

$AC_{\text{loc}}(I, M)$	class of all locally absolutely continuous functions $I \to M$
$L_{\text{loc}}(I, M)$	class of all locally Lebesgue integrable functions $I \to M$
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$.

The subject of our study is the real two-dimensional system

$$x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(\tau(t)) + \mathbf{h}(t, x(t), x(\tau(t))), \tag{0}$$

where $\mathbf{A}(t) = \big(a_{ik}(t)\big)$, $\mathbf{B}(t) = \big(b_{ik}(t)\big)$ $(i, k = 1, 2)$ are real square matrices and $\mathbf{h}(t, x, y) = \big(h_1(t, x, y), h_2(t, x, y)\big)$ is a real vector function. We suppose that the functions $a_{ik} \in AC_{\text{loc}}([t_0, \infty), \mathbb{R})$, $b_{ik} \in L_{\text{loc}}([t_0, \infty), \mathbb{R})$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on

$$[t_0, \infty) \times \{[x_1, x_2] \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < R^2\} \times \{[y_1, y_2] \in \mathbb{R}^2 \colon y_1^2 + y_2^2 < R^2\},$$

where $0 < R \leq \infty$ is a constant and $x = [x_1, x_2]$, $y = [y_1, y_2]$.

For the investigation of the problem we use results obtained by the combination of the method of transformation of the two-dimesional real system into one equation with complex-valued coefficients and the method of Lyapunov-Krasovskii functional.

Introducing complex variables $z = x_1 + ix_2$, $w = y_1 + iy_2$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + A(t)z(\tau(t)) + B(t)\overline{z}(\tau(t)) + g(t, z(t), z(\tau(t))), \tag{1}$$

where $A, B \in L_{\text{loc}}(J, \mathbb{C})$, $a, b \in AC_{\text{loc}}(J, \mathbb{C})$, $g \in K(J \times \Omega, \mathbb{C})$, where $J = [t_0, \infty)$, $\Omega = \big\{(z, w) \in \mathbb{C}^2 \colon |z| < R, |w| < R\big\}$, $R > 0$.

We assume that $\tau \in AC_{\text{loc}}(J, \mathbb{R})$ is such that $\tau(t) \leq t$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

In this presentation we consider the case

$$\liminf_{t\to\infty} \big(|a(t)| - |b(t)|\big) > 0 \tag{2'}$$

and study the behavior of solutions of (1) under this assumption.

The idea is based upon the well known result that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\overline{z}$ ensures that zero is a focus, a centre or a node.

The inequality (2') is equivalent to the existence of $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \geq T_1, \qquad t \geq \tau(t) \geq T_1 \text{ for } t \geq T. \tag{2}$$

Denote $\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}$, $c(t) = \frac{\overline{a}(t)b(t)}{|a(t)|}$.

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality $\gamma(t) > |c(t)| + \mu$ is true for all $t \geq T_1$. It is easy to verify that $\gamma, c \in AC_{\rm loc}([T_1,\infty),\mathbb{C})$.

Further we denote

$$\alpha(t) = 1 + \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t), \quad \vartheta(t) = \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \overline{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}.$$

We will consider following assumptions:

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ are such that (2) holds.

(ii) There are functions $\kappa_0, \kappa_1 \colon [T,\infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t,z,w) + c(t)\overline{g}(t,z,w)| \leq \kappa_0(t)|\gamma(t)z(t) + c(t)\overline{z}(t)| + \kappa_1(t)|\gamma(\tau(t))w + c(\tau(t))\overline{w}|$$

for $t \geq T$, $|z| < R$ and $|w| < R$, where $\kappa_0 \in L_{\rm loc}([T,\infty),\mathbb{R})$.

(iii) $\beta \in AC_{\rm loc}([T,\infty),\mathbb{R}^0_+)$ is a function satisfying $\tau'(t)\beta(t) \geq \psi(t)$ a. e. on $[T,\infty)$, where ψ is defined for every $t \geq T$ by

$$\psi(t) = \kappa_1(t) + (|A(t)| + |B(t)|)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\tau(t))| - |c(\tau(t))|}.$$

(iv) The function $\Lambda \in L_{\rm loc}([T,\infty),\mathbb{R})$ satisfies the inequalities $\beta'(t) \leq \Lambda(t)\beta(t)$, $\theta(t) \leq \Lambda(t)$ for almost all $t \in [T,\infty)$, where the function θ is defined by $\theta(t) = \alpha(t)\operatorname{Re} a(t) + \vartheta(t) + \kappa_0(t) + \beta(t)$.

Notice that from (ii) it follows that the equation (1) has the trivial solution $z(t) \equiv 0$.

Theorem 1. *Let the conditions* (i), (ii), (iii) *and* (iv) *hold. If* $\limsup_{t\to\infty} \int^t \Lambda(s)ds < \infty$, *then the trivial solution of* (1) *is stable on* $[T,\infty)$.

The proof and more results and details are contained in [1].

New results obtained from Theorem 1 are:

Corollary 1. *Let the assumptions* (i), (ii) *and* (iii) *be fulfilled. If for some* $K \in \mathbb{R}_+$ *and* $T_2 \geq T$ *the function* $\beta(t)$ *satisfies* $\beta(T_2) = K$, $\beta(t) \leq K$ *for all* $t \geq T_2$ *and*

$$\lim_{t\to\infty} \int^t [\theta^*(s)]_+ ds < \infty,$$

where $\theta^*(t) = \theta(t) - \beta(t) + K$ *and* $[\theta^*(t)]_+ = \max\{\theta^*(t), 0\}$, *then the trivial solution of* (1) *is stable.*

Corollary 2. *Assume validity of the conditions* (i), (ii) *and* (iii). *If* $\beta(t)$ *is monotone and bounded on* $[T,\infty)$ *and if*

$$\lim_{t\to\infty} \int^t [\theta(s)]_+ ds < \infty,$$

where $[\theta(t)]_+ = \max\{\theta(t), 0\}$, *then the trivial solution of* (1) *is stable.*

References

[1] Kalas J.: *Asymptotic behaviour of a two-dimensional differential systems with nonconstant delay*, submitted for publication.