Stability of the trivial solution of real two-dimensional differential system with nonconstant delay

Josef Rebenda
Masaryk University in Brno, Faculty of Sciences, Department of Mathematics and Statistics
e-mail: rebenda@math.muni.cz

Abstract

In this paper two results from the theory of stability of a real two-dimensional system $x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(\tau(t)) + \mathbf{h}(t, x(t), x(\tau(t))$ are presented, where $\tau(t) \le t$ is nonconstant delay, $\mathbf{A}$, $\mathbf{B}$ are the matrix functions and $\mathbf{h}$ is the vector function.

The results are obtained as consequences of the theorem presented before in the paper by J. Kalas ([1]).

1 Introduction

The following notation will be used throughout the presentation:

$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all positive real numbers
$\mathbb{R}_+^0$	set of all nonnegative real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{N}$	set of all positive integers
$\operatorname{Re} z$	real part of z
$\operatorname{Im} z$	imaginary part of z
$\overline{z}$	complex conjugate of z
$AC_{\text{loc}}(I, M)$	class of all locally absolutely continuous functions $I \to M$
$L_{\text{loc}}(I, M)$	class of all locally Lebesgue integrable functions $I \to M$
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$.

The subject of our study is the real two-dimensional system

$$x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(\tau(t)) + \mathbf{h}(t, x(t), x(\tau(t))), \tag{0}$$

where $\mathbf{A}(t) = \big(a_{ik}(t)\big)$, $\mathbf{B}(t) = \big(b_{ik}(t)\big)$ $(i, k = 1, 2)$ are real square matrices and $\mathbf{h}(t, x, y) = \big(h_1(t, x, y), h_2(t, x, y)\big)$ is a real vector function. We suppose that the functions a_{ik} are locally absolutely continuous on $[t_0, \infty)$, b_{ik} are locally Lebesgue integrable on $[t_0, \infty)$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on

$$[t_0, \infty) \times \{[x_1, x_2] \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < R^2\} \times \{[y_1, y_2] \in \mathbb{R}^2 \colon y_1^2 + y_2^2 < R^2\},$$

where $0 < R \le \infty$ is a constant and $x = [x_1, x_2]$, $y = [y_1, y_2]$.

The aim is to find some sufficient conditions for stability of the trivial solution of equation (0) when this solution exists.

For the investigation of the problem we use results obtained by the combination of the method of transformation of the two-dimesional real system into one equation with complex-valued coefficients and the method of Lyapunov-Krasovskii functional, which is to a great extent effective exactly for two-dimensional systems. This combination was successfully used in [3], [2] and [1] and leads to interesting results.

Introducing complex variables $z = x_1 + ix_2$, $w = y_1 + iy_2$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + A(t)z(\tau(t)) + B(t)\overline{z}(\tau(t)) + g(t, z(t), z(\tau(t)),$$

where

$$\begin{aligned}
a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)),\\
b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)),\\
A(t) &= \frac{1}{2}(b_{11}(t) + b_{22}(t)) + \frac{i}{2}(b_{21}(t) - b_{12}(t)),\\
B(t) &= \frac{1}{2}(b_{11}(t) - b_{22}(t)) + \frac{i}{2}(b_{21}(t) + b_{12}(t)),\\
g(t, z, w) &= h_1(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w + \overline{w}), \frac{1}{2i}(w - \overline{w}))\\
&\qquad + ih_2(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w + \overline{w}), \frac{1}{2i}(w - \overline{w})).
\end{aligned}$$

Conversely, this equation can be written in the real form (0) as well. For details see [2].

2 Results

We study the equation

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + A(t)z(\tau(t)) + B(t)\overline{z}(\tau(t)) + g(t, z(t), z(\tau(t))), \tag{1}$$

where $A, B \in L_{\mathrm{loc}}(J, \mathbb{C})$, $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \Omega, \mathbb{C})$, where $J = [t_0, \infty)$, $\Omega = \{(z, w) \in \mathbb{C}^2 \colon |z| < R, |w| < R\}$, $R > 0$, and $\tau \in AC_{\mathrm{loc}}(J, \mathbb{R})$ is such that $\tau(t) \le t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

In this presentation we consider the case

$$\liminf_{t\to\infty} \big(|a(t)| - |b(t)|\big) > 0 \tag{2'}$$

and study the behavior of solutions of (1) under this assumption.

The idea is based upon the well known result that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\overline{z}$ ensures that zero is a focus, a centre or a node. Details are contained in [3].

The inequality (**??**') is equivalent to the existence of $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \text{ for } t \ge T_1, \qquad t \ge \tau(t) \ge T_1 \text{ for } t \ge T. \tag{2}$$

Denote

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \quad c(t) = \frac{\overline{a}(t)b(t)}{|a(t)|}. \tag{3}$$

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{4}$$

is true for all $t \ge T_1$. It is easy to verify that $\gamma, c \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{C})$.

For the purpose of this paper we denote

$$\alpha(t) = 1 + \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t),$$

$$\vartheta(t) = \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \overline{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}.$$

We will consider following assumptions:

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$ and $\mu > 0$ are such that (2) holds.

(ii) There are functions $\kappa_0, \kappa_1 \colon [T, \infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t, z, w) + c(t)\overline{g}(t, z, w)| \leq \kappa_0(t)|\gamma(t)z(t) + c(t)\overline{z}(t)| + \kappa_1(t)|\gamma(\tau(t))w + c(\tau(t))\overline{w}|$$

for $t \geq T$, $|z| < R$ and $|w| < R$, where $\kappa_0 \in L_{\text{loc}}([T, \infty), \mathbb{R})$.

(iii) $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}^0_+)$ is a function satisfying

$$\tau'(t)\beta(t) \geq \psi(t) \quad \text{a. e. on } [T, \infty),$$

where ψ is defined for every $t \geq T$ by

$$\psi(t) = \kappa_1(t) + (|A(t)| + |B(t)|)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\tau(t))| - |c(\tau(t))|}.$$

(iv) The function $\Lambda \in L_{\text{loc}}([T, \infty), \mathbb{R})$ satisfies the inequalities $\beta'(t) \leq \Lambda(t)\beta(t)$, $\theta(t) \leq \Lambda(t)$ for almost all $t \in [T, \infty)$, where the function θ is defined by

$$\theta(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) + \kappa_0(t) + \beta(t). \tag{5}$$

Clearly, if A, B, κ_1 are absolutely continuous on $[T, \infty)$ and $\psi(t) \geq 0$ on $[T, \infty)$, we may choose $\beta(t) = \psi(t)$.

Under the assumption (i), we can estimate

$$|\vartheta| \leq \frac{|\operatorname{Re}(\gamma\gamma' - \overline{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} = $$
$$= \frac{|\gamma'| + |c'|}{|\gamma| - |c|} \leq \frac{1}{\mu}(|\gamma'| + |c'|),$$

hence the functions ϑ and θ are locally Lebesgue integrable on $[T, \infty)$. Moreover, if $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+)$, then in (iv) we may choose

$$\Lambda(t) = \max\Big(\theta(t), \frac{\beta'(t)}{\beta(t)}\Big).$$

Notice that the condition (ii) implies that the function $\kappa_j(t)$ are nonnegative on $[T, \infty)$ for $j = 0, 1$, and due to this, $\psi(t) \geq 0$ on $[T, \infty)$. Finally, from (ii) it follows that the equation (1) has the trivial solution $z(t) \equiv 0$, which is crucial for our considerations.

Theorem 1. *Let the conditions* (i), (ii), (iii) *and* (iv) *hold.*

a) If

$$\limsup_{t\to\infty} \int^{t} \Lambda(s)ds < \infty,$$

then the trivial solution of (1) *is stable on* $[T, \infty)$*;*

b) if

$$\lim_{t\to\infty}\int^{t}\Lambda(s)ds=-\infty,$$

then the trivial solution of (1) *is asymptotically stable on* $[T,\infty)$.

Proof. The proof is contained in [1]. Remark that the key point of the proof is to find suitable Lyapunov function. It appears to be the function

$$V(t)=U(t)+\beta(t)\int_{\tau(t)}^{t}U(s)ds,$$

where $U(t)=|\gamma(t)z(t)+c(t)\overline{z}(t)|$. □

Remark 1. Since

$$\vartheta=\frac{\operatorname{Re}(\gamma\gamma'-\bar{c}c')+|\gamma c'-\gamma' c|}{\gamma^2-|c|^2}\leq\frac{(|\gamma'|+|c'|)(|\gamma|+|c|)}{\gamma^2-|c|^2}=\frac{|\gamma'|+|c'|}{|\gamma|-|c|},$$

it follows from (4) that we can replace the function ϑ in (5) by $\frac{1}{\mu}(|\gamma'|+|c'|)$.

Now we are able to derive new results from Theorem 1:

Corollary 1. *Let the assumptions* (i), (ii) *and* (iii) *be fulfilled. If for some* $K\in\mathbb{R}_+$ *and* $T_2\geq T$ *the function* $\beta(t)$ *satisfies* $\beta(T_2)=K$, $\beta(t)\leq K$ *for all* $t\geq T_2$ *and*

$$\lim_{t\to\infty}\int^{t}[\theta^*(s)]_+ds<\infty,$$

where $\theta^*(t)=\theta(t)-\beta(t)+K$ *and* $[\theta^*(t)]_+=\max\{\theta^*(t),0\}$, *then the trivial solution of* (1) *is stable.*

Proof. Put

$$\beta^*(t)=\begin{cases}\beta(t) & \text{on } [T,T_2];\\ K & \text{for } t\geq T_2.\end{cases}$$

Then $\beta^*(t)\in AC_{\text{loc}}([T,\infty),\mathbb{R}^0_+)$ and β^* satifies (iii) since $\tau'(t)\beta^*(t)\geq\psi(t)$ a. e. on $[T,\infty)$. Now $(\beta^*)'(t)\equiv 0$ on $[T_2,\infty)$, and also $\frac{(\beta^*)'(t)}{\beta^*(t)}\equiv 0$ on $[T_2,\infty)$. Obviously we may put

$$\Lambda^*(t)=\max\{\theta^*(t),0\}=[\theta^*(t)]_+$$

on $[T_2,\infty)$. Then Λ^* satisfies the condition (iv) on $[T_2,\infty)$ and it follows that

$$\limsup_{t\to\infty}\int^{t}\Lambda^*(s)ds=\limsup_{t\to\infty}\int^{t}[\theta^*(s)]_+ds=\lim_{t\to\infty}\int^{t}[\theta^*(s)]_+ds<\infty.$$

The assertion now follows from Theorem 1. □

Corollary 2. *Assume validity of the conditions* (i), (ii) *and* (iii). *If* $\beta(t)$ *is monotone and bounded on* $[T,\infty)$ *and if*

$$\lim_{t\to\infty}\int^{t}[\theta(s)]_+ds<\infty,$$

where $[\theta(t)]_+=\max\{\theta(t),0\}$, *then the trivial solution of* (1) *is stable.*

Proof. Suppose firstly that β is non-increasing on $[T,\infty)$. Then $\beta' \le 0$ a.e. on $[T,\infty)$.

If $\beta(T_2) = 0$ for some $T_2 \ge T$, then $\beta(t) \equiv 0$ on $[T_2,\infty)$. Consequently, Λ has to satisfy only the inequality $\theta(t) \le \Lambda(t)$ a.e. on $[T_2,\infty)$, so we may choose $\Lambda(t) = \theta(t)$ on $[T_2,\infty)$. It follows that $\Lambda(t) = \theta(t) \le \max\{\theta(t), 0\} = [\theta(t)]_+$.

On the other way, if $\beta(t) > 0$ on $[T,\infty)$, we may put $\Lambda(t) = \max\big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\big\}$. Then

$$\Lambda(t) = \max\Big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\Big\} \le \max\{\theta(t), 0\} = [\theta(t)]_+.$$

In both cases, Λ satisfies the condition (iv) and the inequality $\Lambda(t) \le [\theta(t)]_+$ on $[T_2,\infty)$, hence

$$\limsup_{t\to\infty} \int^t \Lambda(s)ds \le \limsup_{t\to\infty} \int^t [\theta(s)]_+ ds = \lim_{t\to\infty} \int^t [\theta(s)]_+ ds < \infty.$$

Now assume that β is non-decreasing on $[T,\infty)$. Then $\beta' \ge 0$ a.e. on $[T,\infty)$.

If $\beta(t) \equiv 0$ on $[T,\infty)$, we may treat it as above.

Otherwise, there is some $T_3 \ge T$ such that $\beta(t) > 0$ on $[T_3,\infty)$ and we may choose $\Lambda(t) = \max\big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\big\}$ on $[T_3,\infty)$. Clearly Λ satisfies the condition (iv) on $[T_3,\infty)$. Since $\beta' \ge 0$ a.e. on $[T,\infty)$, it follows that $\frac{\beta'}{\beta} \ge 0$ a.e. on $[T_3,\infty)$. Hence

$$\Lambda(t) = \max\Big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\Big\} \le \max\Big\{[\theta(t)]_+, \frac{\beta'(t)}{\beta(t)}\Big\} \le [\theta(t)]_+ + \frac{\beta'(t)}{\beta(t)}$$

and then

$$\limsup_{t\to\infty} \int^t \Lambda(s)ds \le \limsup_{t\to\infty} \int^t [\theta(s)]_+ ds + \limsup_{t\to\infty} \int^t \frac{\beta'(t)}{\beta(t)} ds \le$$
$$\le \lim_{t\to\infty} \int^t [\theta(s)]_+ ds + \limsup_{t\to\infty} \big(\ln(\beta(t))\big) - \ln(\beta(T_3)) < \infty$$

since β is bounded on $[T,\infty)$.

The statement follows from Theorem 1. □

3 Conclusion

We found two sufficient conditions on stability of the trivial solution of real two-dimensional differential system with nonconstant delay. We obtained them from more general sufficient condition. The new results appear to be effective since the assumptions should not be too difficult to verify.

We investigated the case $\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$. Similar and under some conditions more suitable results can be found in the case $\liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0$.

References

[1] Kalas J.: *Asymptotic behaviour of a two-dimensional differential systems with nonconstant delay*, submitted for publication.

[2] Kalas J., Baráková L.: *Stability and asymptotic behaviour of a two-dimensional differential system with delay*, J. Math. Anal. Appl. 269 (2002) 278-300.

[3] Ráb M., Kalas J.: *Stability of dynamical systems in the plane*, Differential Integral Equations 3 (1990) 124-144.