On the (im)possibility of a predator-prey model identification

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1 Introduction

The well-known and widely used model of predator-prey (or producer-consumer, host-parasite, plant-herbivore, panda-bamboo etc.) interaction is the Gause-type one:

$$x' = xq(x) - S\varphi(x)y, \qquad y' = -dy + \kappa S\varphi(x)y,$$

where $x = x(t)$ and $y = y(t)$ denote a size (population density, biomass, number of individuals etc.) of prey and predator populations, respectively, q denotes prey growth rate in an environment with absent predator (it can depend on size of prey population), d denotes constant predator death rate, φ is the so called trophic function (rate of satiety of predator population of unit size fed on prey population of size x in a unit time; 0 – predator is totally starving, 1 – predator is totally saturated), S denotes the level of satiety of predator (the maximal possible size of prey population destroyed by predator population of unit size in a unit time); consequently, $S\varphi(x)$ represent a size of prey population destroyed by a predator population of unit size in unit time, κ denotes efficiency of predation (fraction of destroyed prey population which is transformed into growth of predator population).

The simplest forms of the prey growth rate function are the constant one: $q(x) = r$ (the prey population does not exhibit neither interspecific competition nor mutualism) and the linearly decreasing (Verhulst) one: $q(x) = r(1 - x/K)$ (the prey population exhibits an interspecific competition in an environment with limited resources). Here r denotes an intrinsic growth rate (maximal possible physiological growth rate of population) and K represents the capacity of environment. The constant (i.e. density independent) growth rate can be considered to be a special case of the linearly decreasing one (with $K = \infty$); hence the function q depends on two parameters, $q(x) = q(x; r, K)$.

The trophic function φ should have the following properties:

- $\varphi(0) = 0$ (if none prey is available the predator starves),
- φ is nondecreasing function (if more prey is available the predator is not less satiated),
- $\lim\limits_{x \to \infty} = 1$ (one predator is able to destroy the prey population up to its saturation).

The widely used functions fulfilling the mentioned conditions are

$$\varphi(x) = 1 - \exp(-ax^s) \text{ (the Ivlev type)} \quad \text{and} \quad \varphi(x) = \frac{ax^s}{1 + ax^s} \text{ (the Holling type)},$$

The both functions depend on two parameters, $\varphi(x) = \varphi(x; s, a)$; in the both cases, the parameter s determines the shape of function (the function φ is concave downward if $s \leq 1$ and it is s-shaped otherwise) and a is a kind of scale parameter characterizing "a speed of convergence".

For details and qualitative analysis of the model see e.g. [3, pp. 100–119] or [1, pp. 60–64].

Now, the question arises: Is it possible to distinguish between different growth rate functions and between different trophic functions on the base of field data (observed population densities of prey and predator)? In another words, does field data enable us to identify "the true model" of populations interaction?

2 Results

Having field data, say $(t_0, X_0, Y_0), (t_1, X_1, Y_1), \ldots, (t_n, X_n, Y_n)$ (i.e. population densities of prey X_i and predator Y_i at time instants t_i, $i = 0, 1, \ldots, n$), we can estimate the parameters of a particular model (with definite prey growth rate and/or trophic function); two methods how to do it are described in [2, pp. 12–18]. A goodness of fit can be measured by the euclidian distance of theoretical trajectory (numerical solution of the ODE system with parameters obtained from data) and measured values in logarithmic phase space, $D = \sqrt{\sum_{i=0}^{n} \left[\left(\log \frac{x(t_i)}{X_i} \right)^2 + \left(\log \frac{y(t_i)}{Y_i} \right)^2 \right]}$ (a reasonable assumption is that the observed population densities are log-normal random variables and the solutions of the ODE system are their mean values).

Three types of data were simulated — 1. growth rate with $K = \infty$, φ of the Ivlev type, 2. growth rate with $K < \infty$, φ of the Ivlev type 3. growth rate with $K = \infty$, φ of the Holling type — and sixty "perturbed" data sets were generated (samples of log-normal distribution with the mean value equal to the simulated one and with the coefficient of variation equal to 0.1) three times. For each of the data set, the parameters of model with the genuine and the alternative growth rate or trophic function were estimated and the goodness of fit D was evaluated.

The obtained results are summarized on the table 1. They show, that there is not significant difference in goodness of fit for the right model and for the alternative one. Consequently, neither type of prey growth rate nor type of trophic function can be identified from field data. But the substantial qualitative property of the model (stability of the population steady state) can be established independently on the fact whether the selected model was true or wrong.

		Estimated model		
		$K = \infty$, φ Holling	$K = \infty$, φ Ivlev	$K < \infty$, φ Ivlev
Simulated data	$K = \infty$, φ Holling	$\{0.797, 0.361, 8.251\}$ 100%	$\{1.003, 0.440, 6.762\}$ 100%	
	$K = \infty$, φ Ivlev	$\{1.046, 0.429, 9.820\}$ 100%	$\{1.135, 0.378, 3.302\}$ 96%	$\{1.442, 0.528, 4.621\}$ 100%
	$K < \infty$, φ Ivlev		$\{1.352, 0.826, 4.217\}$ 100%	$\{1.990, 0.761, 12.18\}$ 100%

Table 1: Distance of simulated and estimated trajectories (in the form {median, min, max}) and relative success in establishing stability of equilibrium

References

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