Some remarks on non-integer differential and integral calculus

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Abstrakt

Differential and integral calculus belongs to basic courses of mathematics and everybody understands geometrical and physical meaning of derivative or integral. But, hardly anybody can imagine for example derivative of order 1/2 or even of non-rational order. The branch of mathematics which generalizes calculus to non-integer case is known under the term "fractional calculs" although this name is little bit misleading. Here we give a brief overview of two the most straightforward definitions and some comments on them.

1 Introduction

The fractional calculus means a branch of mathematics dealing with generalization of differential and integral calculus in sense of considering non-integer order derivatives and non-integer-fold integrals. The beginings of the fractional calculus fall to the end of the 17^{th} century, when Leibniz made a note in a correspondence with L'Hospital about meaning of the derivative of order one half. Then, during three centuries, the theory of non-integer derivatives had been developed mainly from a theoretical point of view. Applications appeared just before a few decades, e.g. in material engineering, control of dynamical systems, chemistry and physics.

Several approaches how to define fractional (or non-integer) derivatives (integrals), e.g. Grünwald-Letnikov, Cauchy, Caputo, Riemann-Liouville and others (see e.g. [2], [3],[5],[6]), are known. Last two mentioned are most often used in applications because of relatively weak assumptions on differentiated functions.

Special functions – mainly Euler's Gamma function – play essential role in fractional calculus. We also refer to Beta, Mittag-Leffler, Wright functions that are useful – a brief overview can be found in [6].

In this paper we give a brief overview on approach by Grünwald-Letnikov and Riemann-Liouville approach and also a motivation for these definitions is discussed. Also several examples are added. Note this text is intended for readers that are not familiar with the topic, this is not a rigorous mathematical study and thus the "theorem – proof" style is omitted.

2 Note on binomial formula

In this section we make the first attempt for generalization of integer-order derivative. Let us start with the usual notion of derivative. Derivative of function f at the point x is defined as

$$D^1 f(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}\,.$$

By iterating this formula we obtain (by induction)

$$D^n f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh) \tag{1}$$

for any $n \in \mathbb{N}$, where binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \,.$$

We can see that the sum on the right-hand side is similar to the binomial formula

$$(1-\xi)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \xi^k \tag{2}$$

except the term $f(x-kh)$, which is not in the form ξ^k. To handle this, let us introduce the displacement operator defined as

$$d_h f(x) = f(x-h)\,,$$

of which repeating yields

$$d_h^k f(x) = f(x-kh)\,, \quad k \in \mathbb{N}\,.$$

Then we can rewrite the relation (1) in the form

$$D^n f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} d_h^k f(x) =$$
$$= \lim_{h\to 0} \frac{1}{h^n} (1-d_h)^n f(x) = \lim_{h\to 0} \left(\frac{1-d_h}{h}\right)^n f(x)\,.$$

For convenience, let us further assume that the limit parameter h approaches zero from the right, i.e. we substitute the relations above by $h \to 0+$. The binomial formula (2) can be generalized to a non-integer case as

$$(1-\xi)^p = \sum_{k=0}^{\infty} (-1)^k \binom{p}{k} \xi^k\,,$$

where the coefficient is given by

$$\binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!} = \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)}$$

and the series on the right-hand side is uniformly convergent for $\xi \in [-1,1]$. Taking this relation into account it seems natural to to generalize the n-th derivative to non-integer case by

$$D^p f(x) = \lim_{h\to 0+} \left(\frac{1-d_h}{h}\right)^p f(x) = \lim_{h\to 0+} \frac{1}{h^p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)} f(x-kh)\,, \quad p>0\,. \tag{3}$$

Let us have a look at the limit in (3) more carefully. The reason why we take h positive only is the following. While the computation of integer-order derivative is a local operation, in view of (3) we have to consider $f(\xi)$ to be defined at least on half-axis $\xi \le x$, due to the presence of infinite sum. Relation (3) is so-called left-sided derivative (notation ${}_{-}D^p$ can be used). Similarly we can use right-sided derivative defined

$$\begin{aligned}{}_{+}D^p f(x) &= \lim_{h\to 0-} \frac{1}{h^p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)} f(x-kh) = \\ &= \lim_{h\to 0+} \frac{(-1)^p}{h^p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)} f(x+kh)\,.\end{aligned} \tag{4}$$

In general, existence of the left-sided derivative does not imply existence of the right-sided one and vice versa. Moreover, ${}_{+}D^p f(x)$ does not equal to ${}_{-}D^p f(x)$ until there is a symmetry of the function $f(\xi)$ according to axis $\xi = x$, i.e. $f(\xi - x) = f(x - \xi)$, $\xi \in \mathbb{R}$. It can be proved that the series in the definition converges absolutely and uniformly if f is bounded on $(-\infty, x]$ (in case of right-sided derivative on $[x, \infty)$). Obviously, such condition is very restrictive, e.g. such a simple function as polynomial is not bounded on $(-\infty, x]$. So, one can guess that taking relation (3) as a correct definition of fractional derivative is probably not very convenient. Anyway, for a while let us stay with that and call this "fractional derivative".

The easiest example is the fractional derivative of a constant function, which immediately yields zero function, i.e. $D^p c = 0$, since $\sum_{k=0}^{\infty} (-1)^k \binom{p}{k} = 0$.

Another simple example is derivative of exponential function. Let us consider $f(x) = \exp(cx)$, where $c > 0$ is a real parameter. According to (3) we have

$$
\begin{aligned}
D^p \mathrm{e}^{cx} &= \lim_{h \to 0+} \frac{1}{h^p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+1)}{k! \Gamma(p-k+1)} \mathrm{e}^{c(x-kh)} = \\
&= \mathrm{e}^{cx} \lim_{h \to 0+} \frac{1}{h^p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+1)}{k! \Gamma(p-k+1)} (\mathrm{e}^{-ch})^k \,.
\end{aligned}
$$

Since $|\mathrm{e}^{-ch}| < 1$, the series converges and we can write

$$
D^p \mathrm{e}^{cx} = \mathrm{e}^{cx} \lim_{h \to 0+} \left(\frac{1 - \mathrm{e}^{-ch}}{h} \right)^p = c^p \mathrm{e}^{cx} \,.
$$

This result corresponds well to intuitive expectation. Taking right-sided derivative (4) we would arrive at divergent expression. On the other-hand, for $f(x) = \exp(-cx)$ we get ${}_{+}D^p f(x) = (-c)^p \exp(-cx)$ while (3) diverges.

Now let us consider for for example function $f(x) = x^c$, where c is a real positive parameter. The n-th derivative is

$$
D^n x^c = c(c-1) \ldots (c-n+1) x^{c-n} = \frac{\Gamma(c+1)}{\Gamma(c-n+1)} x^{c-n} \,. \tag{5}
$$

Although evaluation of formula (3) is not possible in this case (function x^c is not bounded on $(-\infty, x]$, in view of (5) it seems that the only reasonable generalization is

$$
\mathcal{D}^p x^c = \frac{\Gamma(c+1)}{\Gamma(c-p+1)} x^{c-p} \tag{6}
$$

(different D symbol is used). Assuming the above considerations are correct, we should not get into troubles when we differentiate exponential function expanded into its Taylor series (such series converges absolutely and uniformly on every closed sub-interval of the real axis). We know that

$$
D^p \mathrm{e}^{cx} = c^p \mathrm{e}^{cx} = c^p \sum_{k=0}^{\infty} \frac{(cx)^k}{k!} = \sum_{k=0}^{\infty} \frac{c^{k+p} x^k}{k!} \,.
$$

On the other hand the differentiation term by term according to (6) yields

$$
D^p \mathrm{e}^{cx} \stackrel{?}{=} \mathcal{D}^p \sum_{k=0}^{\infty} \frac{(cx)^k}{k!} = \sum_{k=0}^{\infty} \frac{c^k x^{k-p}}{\Gamma(k-p+1)} \,.
$$

If we compare both results we can see that they coincide for integer values of p only. This is another "strange result" confirming that there is something wrong in relation (3) to be "good" defintion of the fractional derivative. In the next section we show where this discrepancy comes from. The main reason lies in the fact that the fractional derivatives – as mentioned above – are non-local (relation (3) requires the function f to be defined at least on half-axis!). If function f is defined on an interval $[a, x]$ (in case of right-sided derivative on $[x, b]$) only, one can use the function $\widetilde{f}$ enlarged by zero outside the corresponding interval, i.e.

$$\widetilde{f}(\xi) = \begin{cases} f(\xi)\,, & \xi = [a, x] \\ 0 & \xi < a \end{cases} \quad , \quad \text{or } \widetilde{f}(\xi) = \begin{cases} f(\xi)\,, & \xi = [x, b] \\ 0 & \xi > b \end{cases} \quad .$$

Better way how to take into account the range of evaluation is involved in the concept by Grünwald and Letnikov described in the following section.

3 Grünwald-Letnikov definition

Grünwald-Letnikov derivative (also called Grünwald-Letnikov differintegral) is a similar generalization to the procedure used in binomial formula approach, but it is based on direct modification of relation (1). The essential difference is in handling the summation index n as $h \to 0$. The idea behind is that for $h \to 0$ we let n go to infinity via relation $n = \frac{x-a}{h}$, where a and x are lower and upper terminals of the differentiation. If f is given on $[a, x]$ ($a \in \mathbb{R}$) and $p \in \mathbb{R}$, then the definition reads

$$_aD_x^p f(x) = \lim_{\substack{h \to 0+ \\ nh = x-a}} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)} f(x - kh)\,. \tag{7}$$

To bring more light into this definition let us first take for example $p = 1$ and then $p = -1$. For $p = 1$ we have $\binom{1}{0} = 1$, $\binom{1}{1} = 1$ and $\binom{1}{k} = 0$ for $k = 2, 3, \ldots$. This immediately gives the first-order derivative

$$_aD_x^1 f(x) = \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}\,.$$

Taking $p = -1$, we have $\binom{-1}{k} = \frac{(-1)(-2)\ldots(-1+k+1)}{k!} = (-1)^k$. This yields

$$_aD_x^{-1} f(x) = \lim_{\substack{h \to 0+ \\ nh = x-a}} h \sum_{k=0}^{n} f(x - kh) = \int_a^x f(\xi)\,\mathrm{d}\xi\,,$$

In view of the last relation, Grünwald-Letnikov definition unifies both n-th derivative and n-fold integral as well as it generalizes this case to non-integer one ($p < 0$ means integrals while $p > 0$ means derivatives). This is the reason for notion "differintegral"). More precisely, the conditions on existence of the limit (7) are described by the following:

a) Let f be a continuous function on $[a, x]$ and $p \in \mathbb{R}^+$ be a parameter. Then

$$_aD_x^{-p} f(x) = \frac{1}{\Gamma(p)} \int_a^x (x - \xi)^{p-1} f(\xi)\,\mathrm{d}\xi\,. \tag{8}$$

b) Moreover, if m is an arbitrary integer such that $p < m + 1$ and $f \in C^{m+1}([a, x])$, then we have

$$_aD_x^p f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(x-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^x (x-\xi)^{-p+m} f^{(m+1)}(\xi)\,\mathrm{d}\xi\,. \tag{9}$$

Let us remark that expression (8) for $p \in \mathbb{N}$ is well known Cauchy formula evaluating p-fold integral. The additional assumption $f \in C^{m+1}([a, x])$ from part b) allows us to rewrite (8) using the integration by parts in the form

$$
\begin{aligned}
{}_aD_x^{-p}f(x) &= \frac{1}{\Gamma(p)}\int_a^x (x-\xi)^{p-1}f(\xi)\mathrm{d}\xi = \\
&= \sum_{k=0}^{m}\frac{f^{(k)}(a)(x-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+m+1)}\int_a^x (x-\xi)^{p+m}f^{(m+1)}(\xi)\,\mathrm{d}\xi\,,
\end{aligned}
\tag{10}
$$

so both parts a) and b) can be written in an unified form for $p \in \mathbb{R}$ ($p = 0$ is an identity operator, i.e. the function itself). Let us note that the condition $p < m+1$ makes the integral in (9) convergent (the smallest possible value is determined by $m \leq p < m+1$). The limit in the definition is computed for $h \to 0+$. Similarly, one can also use $h \to 0-$ and the summation bound given by $nh = x + b$ which implies the evaluation range $[x, b]$ (compare to left and right-sided derivative from the previous section). Let us proceed to some examples.

First, the derivative of a constant function is again especially simple and this is left up to the reader (note that derivative of a constant function is not zero!).

Let us consider $f(x) = \mathrm{e}^{cx}$, where $c > 0$ is a real parameter. According to (9) we get

$$
{}_aD_x^p\mathrm{e}^{cx} = \sum_{k=0}^{m}\frac{c^k\mathrm{e}^{ca}(x-a)^{p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)}\int_a^x (x-\xi)^{-p+m}c^{m+1}\mathrm{e}^{c\xi}\mathrm{d}\xi\,,
$$

where $m \in \mathbb{N}$ fulfills $p < m+1$. Substituting $\tau = c(x-\xi)$ we have

$$
{}_aD_x^p\mathrm{e}^{cx} = \sum_{k=0}^{m}\frac{c^k\mathrm{e}^{ca}(x-a)^{p+k}}{\Gamma(-p+k+1)} + \frac{\mathrm{e}^{cx}c^p}{\Gamma(-p+m+1)}\int_0^{c(x-a)} \mathrm{e}^{-\tau}\tau^{-p+m}\mathrm{d}\tau\,.
$$

In particular, considering $a = -\infty$, $p > 0$ and using integral definition of Gamma function we obtain

$$
{}_{-\infty}D_x^p\mathrm{e}^{cx} = c^p\mathrm{e}^{cx}\,.
$$

Note, that the term on the right-hand side coincides with the result obtained via binomial formula in example from previous section. It can be directly seen that using lower terminal $a = -\infty$ in the Grünwald-Letnikov definition we obtain relation (3) from previous section.

Let $f(x) = (x-a)^c$, where c is a real positive parameter such that $c > m$, where $0 \leq m \leq p < m+1$. Then we have

$$
\begin{aligned}
{}_aD_x^p(x-a)^c &= \frac{c(c-1)\cdots(c-m)}{\Gamma(m-p+1)}\int_a^x (x-\xi)^{m-p}(\xi-a)^{c-m-1}\mathrm{d}\xi = \\
&= \frac{\Gamma(c+1)}{\Gamma(c-m)\Gamma(m-p+1)}\int_a^x (x-\xi)^{m-p}(\xi-a)^{c-m-1}\mathrm{d}\xi = \\
&= \frac{\Gamma(c+1)B(m-p+1,c-m)}{\Gamma(c-m)\Gamma(m-p+1)}(x-a)^{c-p} = \frac{\Gamma(c+1)}{\Gamma(c-p+1)}(x-a)^{c-p},
\end{aligned}
$$

where B is the Beta function (see e.g. [1]). Considering $a = 0$ we have

$$
{}_0D_x^px^c = \frac{\Gamma(c+1)}{\Gamma(c-p+1)}x^{c-p}\,.
$$

Now, we can clearly see, why we did not get the expected equality in differentiation of exponential function expanded into its Taylor series. While the derivative of exponential function corresponded to the bound $a = -\infty$, the derivative of power function was in fact carried out for $a = 0$. So, let us again emphasize that the range of evaluation must be taken into account.

4 Riemann-Liouville definition

The approach by Grünwald and Letnikov defined as a limit of a fractional-order backward difference is quite straightforward, but it has some inconveniences. Having a look at the expressions (8) and (9), the first one looks better because of presence of the integral part only. The expression (8) was interpreted as a generalized Cauchy formula for iterated integration. This is the base for Riemann-Liouville definition of a fractional derivative (differintegral). More precisely, let us define integral operator ${}_aJ_x$ by

$$ {}_aJ_x^p f(x) = \frac{1}{\Gamma(-p)} \int_a^x (x-\xi)^{-p-1} f(\xi)\, \mathrm{d}\xi \,. $$

This operator works well for negative values of p, but for $p \geq 0$ integral diverges, so operator ${}_aJ_x$ can not be directly used for definition of fractional derivative. However, this can be overcome very easily. For example if we take $p = 3.2$, we integrate by order -0.8 and then we maintain 4-th order derivative to the result. More precisely Riemann-Liouville definition reads (different symbol $\mathbf{D}$ is used)

$$ {}_a\mathbf{D}_x^p f(x) = \begin{cases} {}_aJ_x^p f(x)\,, & p < 0 \\ f(x)\,, & p = 0 \\ D^n {}_aJ_x{}^{p-n} f(x)\,, & p > 0 \end{cases} . $$

Let us remind that formulas for Grünwald-Letnikov integral and derivative are obtained under the assumption on function $f(x)$ having at least $m+1$ continuous derivatives (m is determined by inequality $p < m+1$). Such condition is quite restrictive although many dynamical processes are smooth enough. On the the other hand Riemann-Liouville definition allows to weaken the conditions on differentiated function. Roughly speaking, it sufficient to assume integrability of $f(x)$ only, differentiation part can be done in sense of generalized derivatives.

The equivalence of both definitions is described by the following proposition:

Suppose that function $f(x)$ is $(n-1)$ times continuously differentiable on $[a, x]$ and $f^{(n)}(x)$ is integrable on $[a, x]$. Then for every $0 < p < n$ Riemann-Liouville derivative coincides with Grünwald-Letnikov derivative, i.e.

$$ {}_a\mathbf{D}_x^p f(x) = {}_aD_x^p f(x)\,. $$

5 Concluding remarks

We have presented some basics of fractional derivatives (integrals). The definition based on generalization of binomial formula is not well posed, but it its understanding explains well, why the summation construction in definition by Grünwald and Letnikov is used. The most common definition of fractional derivative is so-called Riemann-Liouvile derivative (differintegral) which is in fact a generalization of Cauchy formula for computation n-fold integrals. These definitions are equivalent under the assumption of enough smooth function $f(x)$.

We can see that terminal a of the evaluation plays an important role in differentiation as well as it does in integration, i.e. we should have on mind the global behaviour of fractional derivatives. This is also one of the reasons why the fractional calculus had been overlooked for a long time in describing nature. The influence of the lower terminal vanishes in case of $p \in \mathbb{N}$ only.

Yet, we have not discussed properties like composition of fractional derivatives, derivative of composed function, chain rule, change of summation and differentiation in the series, etc. This is left to the readers. As a good starting point we recommend the book [6]. Here we only claim, that these properties are not so apparent as in the integer order calculus.

Besides above mentioned approaches there exist several another ways how to define fractional derivatives. Most of them are equivalent to each other, although some admit wider class of functions or they take some special care on properties at the lower terminal and therefore they are more applicable.

A list of appliactions of the theory can be found e.g. in [4].

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