Numerical Solution of Helmholtz Equation by Boundary Elements Method

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1 Helmholtz Boundary Equation

Space-dependent part of time-harmonic acoustic wave is described by Helmholtz equation

$$\Delta u + k^2 u = 0 \quad k = \frac{\omega}{v}$$

(ω being frequency and v speed of sound in given media). Domain of solution is either interior or exterior of some closed curve (boundary) representing submerged obstacle in 2D. In case of the exterior problem an extra condition must be added to the condition on the body boundary, and it is so called Sommerfeld radiation condition, which ensures the uniqueness of solution to the scattering problem

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|.$$

Applying the second Green's theorem to the problem as stated above we come to the boundary integral form of the Helmholtz equation valid in the interior and the exterior respectively

$$u(x) = \int_\Gamma \left(\frac{\partial u}{\partial \nu_y}(y)\phi(x,y) - u(y)\frac{\partial \phi(x,y)}{\partial \nu_y}\right) ds(y) \quad \text{for} \quad x \in \Omega,$$

and

$$u(x) = \int_\Gamma \left(u(y)\frac{\partial \phi(x,y)}{\partial \nu_y} - \frac{\partial u}{\partial \nu_y}(y)\phi(x,y)\right) ds(y) \quad \text{for} \quad x \in R^n \setminus \bar{\Omega},$$

ν being in both cases the unit normal vector to the boundary pointing into the exterior, $\phi(x,y)$ stands for the fundamental solution (actual form differs in R^2 for R^3). For the limit case of $x \in \partial\Omega$ similar equation holds, namely

$$\int_\Gamma u(y)\frac{\partial \phi(x,y)}{\partial \nu_y} ds(y) - cu(x) = \int_\Gamma \frac{\partial u}{\partial \nu_y}(y)\phi(x,y) ds(y).$$

Both Dirichlet or Neumann boundary value problem can be formulated. Collocation or Galerkin methods can be used for discretization of the boundary integral form of Helmholtz equation. We choose collocation since it is simple to apply. Discretizing boundary into N elements we get

$$\sum_{j=1}^{N} \int_{\Gamma_i} u(y)\frac{\partial \phi(x_i,y)}{\partial \nu_y} ds(y) = c_i u(x_i) + \sum_{j=1}^{N} \int_{\Gamma_i} \frac{\partial u}{\partial \nu_y}(y)\phi(x_i,y) ds(y).$$

2 Burton-Miller Method

Numerical experiments have shown, that the standard approach to solution described above only gives reliable results for limited range of wave numbers, approximately up to $kD < 4$, where D stands for characteristic dimension of the body. Therefore, effort has been devoted to development of improved methods which would allow to overcome this major drawback. The most promising appears to be Burton-Miller scheme with hybrid equation

$$\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y) - cu(x) + \mu\frac{\partial}{\partial\nu_x}\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y) =$$
$$= \int_\Gamma \frac{\partial u}{\partial\nu_y}(y)\phi(x,y)ds(y) + \mu\frac{\partial}{\partial\nu_x}\int_\Gamma \frac{\partial u}{\partial\nu_y}(y)\phi(x,y)ds(y) + c\frac{\partial u}{\partial\nu_y}(y) \quad \text{for} \quad x \in \Gamma.$$

While majority of the calculation steps remain the same, the difficulty of solving Burton-Miller problem lays in the numerical evaluation of $\frac{\partial}{\partial\nu_x}\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y)$, for which a special procedure must be developed.

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