Numerical Solution of Helmholtz Equation by Boundary Elements Method

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Abstract

Boundary elements method is well-suited for computational acoustics. The main profit it brings is reduction in number of unknowns compared to FEM. However, application of BEM to sound scattering problems has shown, that this advantage is lost in case of high frequency waves. High frequencies require larger number of elements which contradicts the initial purpose of using BEM in solvers and may even result in situations, where solution does not exist for some exterior problems. A special choice of basis functions of exponential type can help to reduce computational cost for high wave numbers. The standard boundary integral formulation can further be improved using Burton-Miller scheme to avoid problems with existence of solution.

1 Helmholtz Boundary Equation

Space-dependent part of time-harmonic acoustic wave is described by Helmholtz equation

$$\Delta u + k^2 u = 0 \quad k = \frac{\omega}{v}$$

(ω being frequency and v speed of sound in given media). Domain of solution is either interior or exterior of some closed curve (boundary) representing submerged obstacle in 2D. In case of the exterior problem an extra condition must be added to the condition on the body boundary, and it is so called Sommerfeld radiation condition, which ensures the uniqueness of solution to the scattering problem

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|.$$

Applying the second Green's theorem to the problem as stated above we come to the boundary integral form of the Helmholtz equation valid in the interior and the exterior respectively

$$u(x) = \int_\Gamma \left(\frac{\partial u}{\partial \nu_y}(y)\phi(x,y) - u(y)\frac{\partial \phi(x,y)}{\partial \nu_y}\right) ds(y) \quad \text{for} \quad x \in \Omega,$$

and

$$u(x) = \int_\Gamma \left(u(y)\frac{\partial \phi(x,y)}{\partial \nu_y} - \frac{\partial u}{\partial \nu_y}(y)\phi(x,y)\right) ds(y) \quad \text{for} \quad x \in R^n \setminus \bar{\Omega},$$

ν being in both cases the unit normal vector to the boundary pointing into the exterior, $\phi(x, y)$ stands for the fundamental solution (actual form differs in R^2 for R^3). For the limit case of $x \in \partial\Omega$ similar equation holds, namely

$$\int_\Gamma u(y)\frac{\partial \phi(x,y)}{\partial \nu_y} ds(y) - cu(x) = \int_\Gamma \frac{\partial u}{\partial \nu_y}(y)\phi(x,y)ds(y).$$

Both Dirichlet or Neumann boundary value problem can be formulated. The problem of greatest interest is a Dirichlet problem with plane wave $u = Ae^{-ikdx}$, where d stands for direction, as the incident field. It represents sound-scattering of acoustic waves with applications in sonars or industrial noise reduction. Since we only study two-dimensional cases, the fundamental solution will take form

$$\phi(x,y) = \frac{i}{4}H_0^1(kr).$$

Parameter c is very often given as a constant of value 0.5. In fact, this only holds for C^1 smooth boundaries, [**?**]. For boundaries with corners, c is a function of the interior angle at a given point

$$c = \frac{\theta}{2\pi}.$$

Collocation or Galerkin methods can be used for discretization of the boundary integral form of Helmholtz equation. We choose collocation since it is simple to apply. Discretizing boundary into N elements we get

$$\sum_{j=1}^{N} \int_{\Gamma_i} u(y)\frac{\partial \phi(x_i,y)}{\partial \nu_y} ds(y) = c_i u(x_i) + \sum_{j=1}^{N} \int_{\Gamma_i} \frac{\partial u}{\partial \nu_y}(y)\phi(x_i,y)ds(y).$$

Using polynomial (constant, piece-wise linear, quadratic) basis functions for approximation of u or $\frac{\partial u}{\partial \nu_y}$ we obtain system of linear equations for the unknown values at nodes x_i. However, with increasing frequency of the incident wave the number of elements grows fast and the reliability of solution drops. Therefore, special basis functions in the form $p(x)e^{-ikdx}$, where p is polynomial, have been suggested, [**?**],[**?**]. In theory, adoption of such basis functions should lead to drastic decrease of the number of elements needed, which is our goal to prove or reject.

2 Numerical Results

So far we have developed solver implementing collocation method for discretization of the boundary integral, using standard 1st and 2nd order polynomial basis functions. Collocation method suitably handles some computational aspects, namely the integration of singular Hankel functions. Galerkin approach does not seem to give substantialy better results, while it is more expensive due to the need of numerical evaluation of double integrals (again with singular kernels). Numerical solution is presented for exterior problem with constant boundary condition.

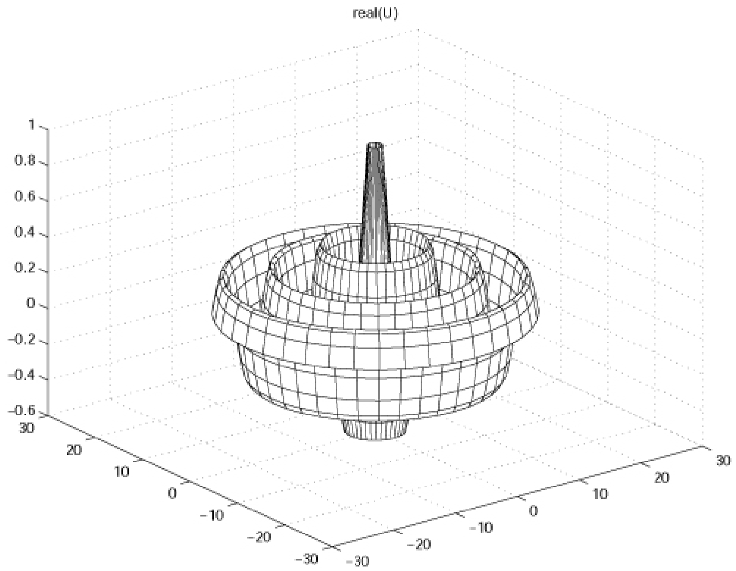

Figure 1: Constant boundary condition - real part of solution.

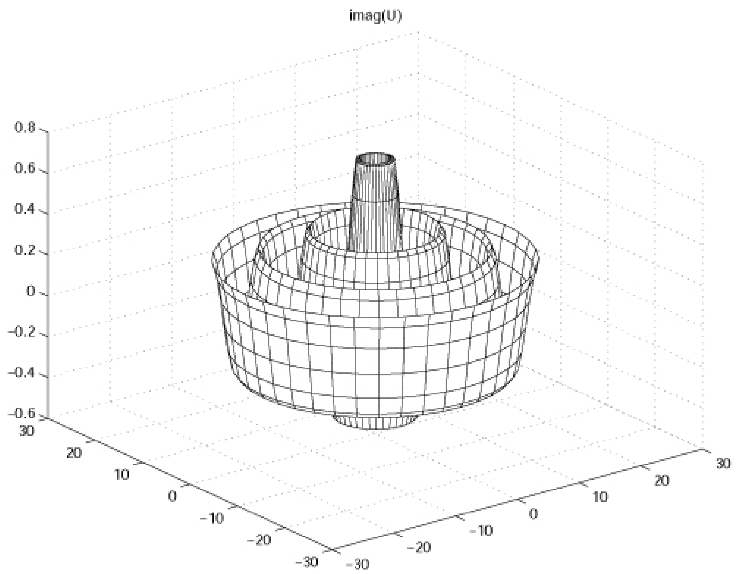

Figure 2: Constant boundary condition - imaginary part of solution.

3 Burton-Miller Method

Numerical experiments have shown, that the standard approach to solution described above only gives reliable results for limited range of wave numbers, approximately up to $kD < 4$, where D stands for characteristic dimension of the body, [**?**]. Therefore, effort has been devoted to development of improved methods which would allow to overcome this major drawback. The most promising appears to be Burton-Miller scheme with hybrid equation

$$\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y) - cu(x) + \mu\frac{\partial}{\partial\nu_x}\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y) =$$
$$= \int_\Gamma \frac{\partial u}{\partial\nu_y}(y)\phi(x,y)ds(y) + \mu\frac{\partial}{\partial\nu_x}\int_\Gamma \frac{\partial u}{\partial\nu_y}(y)\phi(x,y)ds(y) + c\frac{\partial u}{\partial\nu_y}(y) \quad \text{for} \quad x \in \Gamma.$$

While majority of the calculation steps remain the same, the difficulty of solving Burton-Miller problem lays in the numerical evaluation of $\frac{\partial}{\partial\nu_x}\int_\Gamma u(y)\frac{\partial\phi(x,y)}{\partial\nu_y}ds(y)$, for which a special procedure must be developed.

4 Conclusions

It turns out that for effective solution of the whole range of interior and exterior problems with arbitrary frequencies the improved Burton-Miller scheme has to be implemented into the solver. This should further by supplemented by adoption of exponential basis functions to obtain high rates of convergence for the practically relevant cases of plane waves as the incident fields. This is the goal of our future work.

Acknowledgement. This research has been supported by the Grant of Czech Academy of Sciences 106/05/2731.

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