Iterations in the space of strictly monotonic functions

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The classical theory of k-th order linear functional and difference equations is obtained as a special case of the theory developed for the k-th order functional equation model which generalizes the first order model $f \circ \Phi - Af = B$. The function Φ is assumed to solve a given Abel functional equation and belong to a special space S of continuous strictly monotonic functions equipped with a group multiplication $\circ$. The theory developed applies to solve the k-th order linear equation using roots of the characteristic equation together with a continuous solution α of the associated Abel functional equation $\alpha \circ \Phi(x) = X(x+1) \circ \alpha(x)$, see [1], [2], [4]. Illustrations of iterative solutions formulas are given for the first order linear equation $f \circ \Phi(x) = Af(x) + B$. The formulas produce approximate values for the solution $f(x)$ and aid in computer assisted visualization of solutions, see [3].

Here the newly defined terms of composition of functions (multiplication) and iterations of a function in the space S of strictly monotonic functions and some properties and also various examples to illustrate the terms are shown.

Definition 1 Let $a, b \in \bar{\mathsf{R}}$, $a < b$, $\mathcal{J} = (-\infty, \infty)$. The set of all functions f which satisfy $f \in C_0(\mathcal{J})$ and f maps one-to-one the interval $\mathcal{J}$ on the interval (a, b) will be denoted by the symbol S_a^b and called a *space of strictly monotonic functions.*

Definition 2 An arbitrarily chosen increasing function $X = X(x)$, $X \in S$, will be called a *canonical function* in S.

Definition 3 Let $\alpha, \beta \in S$. Let X^* be the inverse to the canonical function $X \in S$. A function $\gamma = \gamma(x)$ defined by

$$\gamma = \alpha X^* \beta(x),$$

where the expression on the right side is a composite function $\alpha[X^*(\beta(x))]$ defined on $\mathcal{J}$, will be called a *product* γ of functions $\alpha, \beta \in S$ in the class S and we write $\gamma = \alpha \circ \beta$.

Definition 4 Let $X \in S$ be the canonical function. Let $\Phi \in S$. The *iterations* of a function Φ in S are given by

$$\Phi^0(x) = X(x)$$
$$\Phi^{n+1}(x) = \Phi \circ \Phi^n(x), \quad x \in \mathcal{J}, \quad n = 0, 1, 2, \dots$$
$$\Phi^{n-1}(x) = \Phi^{-1} \circ \Phi^n(x), \quad x \in \mathcal{J}, \quad n = 0, -1, -2, \dots,$$

where $\Phi^{-1} = \hat{\Phi}$ is the inverse element to the element Φ in S according to the multiplication $\circ$.

Definition 5 Let $X \in S$ be a canonical function. Let $\phi, \Phi \in S$, $\phi > X$, $\Phi > X$. Functional equation

$$\alpha \circ \Phi(x) = \phi \circ \alpha(x)$$

is called an *equation of Abel's type* in S.

Theorem 1 Assume $\alpha, g, X \in S$ and X is a canonical function. Let α be a solution of the Abel functional equation

$$\alpha \circ g = X(x+1) \circ \alpha.$$

Then the function

$$\Phi = \hat{\alpha} \circ X\left(x + \frac{1}{n}\right) \circ \alpha(x),$$

where $\hat{\alpha}$ is the inverse to α, satisfies equation

$$\Phi^n = g(x), \quad n \in \mathsf{N}.$$

References

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