Iterations in the space of strictly monotonic functions

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August 2007

Abstract

A generalization of the classical theory of linear difference equations and linear functional equations and an unifying view on those theories can be built in the space S of strictly monotonic functions. That is enabled by introducing generalized terms of composition of functions, iterations of a function in S under group multiplication, difference of a function in S and others.
In the paper iterations of a function in the space S of strictly monotonic functions are investigated. The classical definition of iterations can be obtained as a special case of the new definition. The newly defined iterations are used in the case of functional equations of Abel's type to illustrate the terms and their properties. In the paper various examples are solved.

1 Introduction

A generalized theory of linear difference and functional equations based upon generalized terms of composition of functions and iterations of a function in the space S of strictly monotonic functions can be built and its applications can be shown, see [1], [2], [3], [4] and [5].

In the paper we define newly iterations of a function in S and we study their properties. Our considerations are done in the space of real-valued functions of a real variable $x \in \mathcal{J}$, $\mathcal{J} = (-\infty, \infty)$. N denotes the set of positive integers, Z integers, R real numbers, $N_0 = N \cup \{0\}$, $\bar{\mathsf{R}} = \langle -\infty, \infty \rangle$. $C_0(\mathcal{J})$ denotes the set of continuous functions on the interval $\mathcal{J}$.

2 Definition of the space S

Definition 1 Let $a, b \in \bar{\mathsf{R}}$, $a < b$. The set of all functions f which satisfy

(1) $f \in C_0(\mathcal{J})$,

(2) f maps one-to-one the interval $\mathcal{J}$ on the interval (a, b).

will be denoted by the symbol S_a^b and called a *space of strictly monotonic functions.*

The set S_a^b is abbreviated to S, whenever possible.

Note 1 Thus only all continuous functions which are increasing from a to b or decreasing from b to a on the interval $\mathcal{J}$ belong to the set S_a^b.

Definition 2 An arbitrarily chosen increasing function $X = X(x)$, $X \in S$, will be called a *canonical function* in S. The inverse to the canonical function X will be denoted by X^*.

Note 2 The domain of the function X^* is the interval (a, b).

3 Multiplication in S

The function $X^* = X^*(x)$, which is the inverse to the canonical function X, is used to define a multiplication of elements in S.

Definition 3 Let $\alpha, \beta \in S$. Let X^* be the inverse to the canonical function $X \in S$. A function $\gamma = \gamma(x)$ defined by

$$\gamma = \alpha X^* \beta(x),$$

where the expression on the right side is a composite function $\alpha[X^*(\beta(x))]$ defined on $\mathcal{J}$, will be called a *product* γ of functions $\alpha, \beta \in S$ in the class S and we write $\gamma = \alpha \circ \beta$.

Lemma 1 If $\alpha, \beta \in S$, then $\gamma \in S$, where $\gamma = \alpha \circ \beta = \alpha X^* \beta$.

Proof It is easy to see that the function γ maps one-to-one the interval $\mathcal{J}$ onto the interval (a, b). $\square$

Lemma 2 The operation of multiplication $\circ$ in S is associative.

Proof A composition of functions is associative. $\square$

Lemma 3 The canonical function X is a neutral element for the multiplication in S.

Proof $X \circ \alpha = \alpha \circ X = \alpha$ holds. $\square$

Lemma 4 To an element $\alpha \in S$ there is an inverse element $\hat{\alpha} = X\alpha^* X$ in S, where $X\alpha^* X$ denotes a composite function $X(\alpha^*(X))$ in S.

Proof It is $\hat{\alpha} = X\alpha^* X$, where α^* is the inverse function to the function α, as

$$\alpha \circ \hat{\alpha} = \alpha X^* X \alpha^* X(x) = X(x), \quad \hat{\alpha} \circ \alpha = X\alpha^* X X^* \alpha(x) = X(x).$$

$\square$

Lemmas 1, 2, 3 and 4 imply the following consequence.

Corollary The set S with the operation of multiplication $\circ$ forms a non-commutative group.

Lemma 5 Let α, β. $\Phi \in S$. Then

$$(\alpha + \beta) \circ \Phi = \alpha \circ \Phi + \beta \circ \Phi,$$

$$(\alpha\beta) \circ \Phi = (\alpha \circ \Phi)(\beta \circ \Phi),$$

$$(\frac{\alpha}{\beta}) \circ \Phi = \frac{\alpha \circ \Phi}{\beta \circ \Phi},$$

if the fractions are defined.

Proof Composite function rules imply the results.

4 Multiplication of functions in $C_0(\mathcal{J})$ by functions from S

Definition 4 Let $f \in C_0(\mathcal{J})$, $\alpha \in S$. The composite function

$$f \circ \alpha = f X^* \alpha(x),$$

where X^* is the inverse to the canonical function $X \in S$ will be called a *product* $f \circ \alpha$.

Lemma 6 If $f \in C_0(\mathcal{J})$, $\alpha \in S$, then $f \circ \alpha \in C_0(\mathcal{J})$.

Iterations in S

Introduced here is the notion of iteration of a function Φ in S under group multiplication.

Definition 5 Let $X \in S$ be the canonical function. Let $\Phi \in S$. The *iterations* of a function Φ in S are given by

$$\Phi^0(x) = X(x) \tag{1}$$

$$\Phi^{n+1}(x) = \Phi \circ \Phi^n(x), \quad x \in \mathcal{J}, \quad n = 0, 1, 2, \ldots \tag{2}$$

$$\Phi^{n-1}(x) = \Phi^{-1} \circ \Phi^n(x), \quad x \in \mathcal{J}, \quad n = 0, -1, -2, \ldots, \tag{3}$$

where $\Phi^{-1} = \hat{\Phi}$ is the inverse element to the element Φ in S according to the multiplication $\circ$.

Note 3 Here and further in the text the exponent n of the function $\Phi^n(x)$ denotes the n-th group multiplication iterate of the function Φ.

Note 4 Formula (3) for $n = 0$ implies

$$\Phi^{-1}(x) = \hat{\Phi}(x),$$

that is $\Phi^{-1}(x)$ is the inverse element to $\Phi(x)$ in S.

Note 5 If $\Phi(x) > X(x)$ in definition 5, then iterates $\Phi^n(x)$ form a monotonic sequence.

Lemma 7 All iterations of the function $\Phi \in S$ belong to S.

Proof Lemma 1 applies. □

Lemma 8 Let μ, $\nu \in$ Z. Then

$$\Phi^\mu \circ \Phi^\nu = \Phi^{\mu+\nu}.$$

Proof Composite function rules apply. □

Abel functional equation in S.

Definition 6 Let $X \in S$ be a canonical function. Let $\phi, \Phi \in S$, $\phi > X$, $\Phi > X$. Functional equation

$$\alpha \circ \Phi(x) = \phi \circ \alpha(x) \tag{4}$$

is called an *equation of Abel's type* in S. The unknown function α is expected to be in S.

If we set $\phi(x) = X(x+1)$ in (4), where $X \in S$ is the canonical function, then equation (4) is of the form

$$\alpha \circ \Phi(x) = X(x+1) \circ \alpha(x) \tag{5}$$

and it is called an *Abel functional equation* in S.

Note 6 It is easy to see that functions ϕ, Φ in (4) are at the same time both either increasing or decreasing.

Lemma 9 If functional equation of Abel's type (4) holds, then

$$\alpha \circ \Phi^{\mu}(x) = \phi^{\mu} \circ \alpha(x), \quad \mu \in \mathsf{Z}, \tag{6}$$

where ϕ^{μ} and Φ^{μ} are μ-th iterates in S of ϕ and Φ, respectively.

Proof. If we multiply repeatedly equation (4) from the right side by the function Φ, then after rearrangement we get the assertion for positive μ. Equivalent to equation (4) is the equation

$$\alpha \circ \Phi^{-1}(x) = \phi^{-1} \circ \alpha(x). \tag{7}$$

If we multiply this equation repeatedly from the right side by the function $\Phi^{-1}(x)$, then we get the assertion for negative μ.

Lemma 10 Let $\mu \in \mathsf{Z}$. If (6) and

$$\alpha \circ \Phi^{\mu-1}(x) = \phi^{\mu-1} \circ \alpha(x) \tag{8}$$

holds for ϕ, $\Phi \in S$, then

$$\alpha \circ \Phi(x) = \phi \circ \alpha(x).$$

Proof. Multiplying equation (8) from the right side by the function $\Phi^{-\mu+1}$ and from the left by $\phi^{-\mu+1}$ we get

$$\alpha \circ \Phi^{-\mu+1}(x) = \phi^{-\mu+1} \circ \alpha(x). \tag{9}$$

If we multiply now equation (6) from the right by the function $\Phi^{-\mu+1}$ then we get

$$\alpha \circ \Phi(x) = \phi^{\mu} \circ \alpha \circ \Phi^{-\mu+1}(x).$$

Using the equation (9) we have

$$\alpha \circ \Phi(x) = \phi^{\mu} \circ \phi^{-\mu+1} \circ \alpha(x) = \phi \circ \alpha(x). \quad \square$$

Now we will deal with the Abel functional equation (5). Because

$$X(x+1) \circ \alpha = X(x+1)X^*\alpha = X(X^*\alpha + 1),$$

then, after a rearrangement, equation (5) can be written in the form

$$\alpha \circ \Phi = X(X^*\alpha + 1), \tag{10}$$

where X^* is the inverse to canonical function $X = X(x)$, $X \in S$.

Note 7 Equation (10) yields

$$\alpha \circ \Phi^2 = X(X^*\alpha + 1) \circ \Phi = X(X^*\alpha \circ \Phi + 1) = X(X^*X(X^*\alpha + 1) + 1) = X(X^*\alpha + 2).$$

In general,

$$\alpha \circ \Phi^{\mu}(x) = X(X^*\alpha + \mu), \quad \mu \in \mathsf{Z}.$$

Theorem 1 Let $\Phi \in S_{-\infty}^{+\infty}$. If $\alpha \in S_{-\infty}^{+\infty}$ is a solution of the Abel functional equation

$$\alpha\Phi(x) = \alpha(x) + 1, \tag{11}$$

then $\beta = X\alpha \in S_{-\infty}^{+\infty}$ is a solution of the Abel functional equation

$$\beta \circ \Psi(x) = X(x+1) \circ \beta, \tag{12}$$

where $\Psi = X\Phi$, $X \in S$ a canonical function.

Proof Assume (11) holds. If we substitute for $\beta = X\alpha$ in (12), then we get

$$X\alpha X^* X\Phi(x) = X(x+1)X^*X\alpha$$

or

$$X\alpha\Phi(x) = X(\alpha(x) + 1)$$

and then $\alpha\Phi(x) = \alpha(x) + 1$, which is valid according to (11). $\square$

4.1 Examples of functions $\Phi \in S_{-\infty}^{+\infty}$, $\Phi(x) > x$

Example 1 $\Phi(x) = x + k, \quad k > 0.$

Indeed, function $\Phi > x$ and Φ increases from $-\infty$ to ∞.

Example 2 $\Phi(x) = \frac{1}{1+x^2} + x.$

Again, $\Phi > x$. We will show that Φ increases from $-\infty$ to $+\infty$.

$$\Phi'(x) = -2x(1+x^2)^{-2} + 1 = \frac{x^4 + 2x^2 + 1 - 2x}{(1+x^2)^2} = \frac{x^4 + x^2 + (x-1)^2}{(1+x^2)^2} > 0.$$

$$\lim_{x\to-\infty} \Phi(x) = \lim_{x\to-\infty} \frac{1+x+x^3}{1+x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}+\frac{1}{x}+x}{\frac{1}{x^2}+1} = -\infty,$$
$$\lim_{x\to\infty} \Phi(x) = \lim_{x\to\infty} \frac{1+x+x^3}{1+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}+\frac{1}{x}+x}{\frac{1}{x^2}+1} = \infty.$$

Thus $\Phi \in S_{-\infty}^{\infty}$.

Example 3 $\Phi(x) = e^{-x^2} + x.$

We observe that $\Phi > X$. It will be shown that Φ increases from $-\infty$ to ∞. We use the well-known identity

$$e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots.$$

Then

$$e^{x^2} - 2x = (1-x)^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots > 0.$$

Thus $e^{x^2} + x > 3x > x$ in $\mathcal{J}$ and

$$\lim_{x\to-\infty} (e^{-x^2} + x) = 0 - \infty = -\infty, \quad \lim_{x\to\infty} (e^{-x^2} + x) = 0 + \infty = \infty.$$

Example 4 $\Phi(x) = \sqrt[2n+1]{x^{2n+1} + 1} \in S_{-\infty}^{\infty}$, $n = 1, 2, 3, \ldots$.

Note 8 Assume the Abel functional equation in $S_{-\infty}^{\infty}$ is given as

$$\alpha\Phi(x) = \alpha(x) + 1. \tag{13}$$

If $\Phi = \sqrt[3]{x^3 + 1}$ then $\alpha = x^3$ is a solution of equation (13) and $\Phi(x) > x$ holds.
If $\Phi = x + \frac{1}{a}$, $a > 0$, then $\alpha(x) = ax + b$, $a > 0$, and $\Phi(x) > x$ holds.

5 Iterates $\{\Phi^\mu(x)\}$ of a function $\Phi \in S$

Theorem 2 Assume $\Phi, X \in S$, X is a canonical function, $\Phi(x) > X(x)$ for $x \in \mathcal{J}$ and $\Phi^\mu(x)$ is the μ-th iterate of Φ in S. Then

1. $\Phi^n(x) > \Phi^{n-1}(x) > X(x), \quad n = 2, 3, \dots, \quad x \in \mathcal{J},$
2. $\Phi^{-n-1}(x) < \Phi^{-n} < X(x), \quad n = 1, 2, \dots, \quad x \in \mathcal{J}.$

Proof

First we prove 1.

Inequality $\Phi(x) > X(x)$, $x \in \mathcal{J}$, is valid also for argument $X^*\Phi(x)$, thus we have

$$\Phi X^* \Phi(x) > X X^* \Phi(x) = \Phi(x) > X(x)$$

or

$$\Phi^2(x) > \Phi(x) > X(x), \quad x \in \mathcal{J}.$$

Using mathematical induction we will prove the following relation:

$$\Phi^n(x) > \Phi^{n-1}(x) > X(x), \quad n = 2, 3, \dots. \tag{14}$$

It is valid for $n = 2$. Assume it is valid for $n \geq 2$. We will show that it is valid for $n+1$. If we set $x = X^*\Phi(x)$ in (14) we get

$$\Phi^{n+1} > \Phi^n(x) > \Phi(x) > X(x), \quad x \in \mathcal{J}.$$

Now we prove 2.

Let Φ^* be the inverse function to the function $\Phi(x)$. Then

$$x = \Phi\Phi^*(x), \quad x \in (a, b)$$

and the inequality $\Phi(x) > X(x)$ implies

$$x = \Phi\Phi^*(x) > X\Phi^*(x), \quad x \in (a, b).$$

This inequality is valid also for the argument $X(x), x \in \mathcal{J}$, thus

$$X(x) > X\Phi^* X(x) = \hat{\Phi}(x),$$

or

$$\Phi^{-1} = \hat{\Phi}(x) < X(x), x \in \mathcal{J}.$$

For $n = -2$ we have

$$\Phi^{-2} = \Phi^{-1} \circ \Phi^{-1} = \hat{\Phi} \circ \hat{\Phi} = X\Phi^* X X^{-1} \hat{\Phi} = X\Phi^* \hat{\Phi} < X\Phi^* X = \hat{\Phi} = \Phi^{-1} < X,$$

because the function $X\Phi^*$ is increasing. To prove the inequality we will use mathematical induction. Let for $n \geq 2$

$$\Phi^{-n}(x) < \Phi^{-n+1}(x) < X(x), \quad x \in \mathcal{J}.$$

We will prove the validity of the inequality for $n+1$. Let $x = X^* \Phi^{-1}(x)$, where $\Phi^{-1} = \hat{\Phi}$, $x \in \mathcal{J}$, then

$$\Phi^{-n} X^* \Phi^{-1} < \Phi^{-n+1} X^* \Phi^{-1} < X X^* \Phi^{-1}$$

or

$$\Phi^{-n-1}(x) < \Phi^{-n}(x) < \Phi^{-1}(x) = \hat{\Phi} < X(x).$$

Corollary Let $X \in S$ be a canonical function and $\Phi \in S$ be an increasing function. Let $\Phi(x) > X(x)$. Then the two-sided sequence of functions $\{\Phi^{\mu}(x)\}_{\mu=-\infty}^{+\infty}$, where Φ^{μ} is an iterate of the function Φ in S, is an increasing sequence.

6 Iterate equations

Theorem 3 Assume $\alpha, g, X \in S$ and X is a canonical function. Let α be a solution of the Abel functional equation

$$\alpha \circ g = X(x+1) \circ \alpha. \tag{15}$$

Then the function

$$\Phi = \hat{\alpha} \circ X\left(x + \frac{1}{n}\right) \circ \alpha(x) \tag{16}$$

satisfies equation

$$\Phi^n = g(x), \quad n \in \mathsf{N}. \tag{17}$$

Proof According to (16)

$$\begin{aligned} \Phi^2 = \Phi \circ \Phi &= \hat{\alpha} \circ X\left(x + \frac{1}{n}\right) \circ \alpha(x) \circ \hat{\alpha} \circ X\left(x + \frac{1}{n}\right) \circ \alpha(x) \\ &= \hat{\alpha} \circ X\left(x + \frac{2}{n}\right) \circ \alpha(x), \end{aligned}$$

because

$$X\left(x+\frac{1}{n}\right)\circ X\left(x+\frac{1}{n}\right)=X\left(x+\frac{2}{n}\right).$$

The proof will proceed by mathematical induction. Let

$$\Phi^{n-1}=\hat{\alpha}\circ X\left(x+\frac{n-1}{n}\right)\circ\alpha(x)$$

hold for $n-1$, then

$$\begin{aligned}\Phi^{n} &= \Phi\circ\Phi^{n-1}\\ &= \hat{\alpha}\circ X\left(x+\frac{1}{n}\right)\circ\alpha(x)\circ\hat{\alpha}\circ X\left(x+\frac{n-1}{n}\right)\circ\alpha(x)\\ &= \hat{\alpha}\circ X(x+1)\circ\alpha\\ &= \hat{\alpha}\circ\alpha\circ g=g.\end{aligned}$$

The following theorem can be proved similarly.

Theorem 4 Assume $\alpha, g, X\in S$ and X is a canonical function. Let α be a solution of the Abel functional equation

$$\alpha\circ g=X(x-1)\circ\alpha.$$

Then the function

$$\Phi=\hat{\alpha}\circ X\left(x-\frac{1}{n}\right)\circ\alpha(x)$$

satisfies equation

$$\Phi^{n}=g(x),\quad n\in\mathsf{N}.$$

Especially, let $S=S_{-\infty}^{\infty}$, $X=x$, $g,\alpha\in S_{-\infty}^{\infty}$. Let α be a solution of the Abel functional equation

$$\alpha g(x)=\alpha(x)+1.$$

Then the function

$$\Phi=\alpha^{*}\left[\alpha(x)+\frac{1}{n}\right],$$

where α^{*} is the inverse function to α, satisfies the equation

$$\Phi^{n}=g(x),\quad n\in\mathsf{N}.$$

Example 5 Prove the following assertion. If

$$\Phi = \tan\left(\arctan x + \frac{\pi}{n}\right), \tag{18}$$

then

$$\Phi^n = x, \quad n \in \mathsf{N},$$

where Φ^n is n-th iteration of the function Φ.

Solution

$$\begin{aligned}\Phi^2(x) &= \tan\left(\arctan\Phi(x) + \frac{\pi}{n}\right)\\ &= \tan(\arctan x + \frac{\pi}{n} + \frac{\pi}{n})\\ &= \tan\left(\arctan x + \frac{2\pi}{n}\right).\end{aligned}$$

Assume

$$\Phi^{n-1}(x) = \tan\left(\arctan x + \frac{n-1}{n}\,\pi\right)$$

holds for $n-1$, then by mathematical induction we get for $x \in \mathcal{J}$

$$\begin{aligned}\Phi^n(x) &= \Phi\Phi^{n-1}(x)\\ &= \tan(\arctan\Phi^{n-1}(x) + \frac{n-1}{n}\pi\\ &= \tan(\arctan x + \frac{n-1}{n}\pi + \frac{1}{n}\pi)\\ &= \tan(\arctan x + \pi)\\ &= \tan\arctan x\\ &= x.\end{aligned}$$

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