An Application of the Kalman-Bucy filter to electrical circuits

Edita Kolářová
Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics
e-mail: kolara@feec.vutbr.cz

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1 Introduction

This paper deals with the filtering problem, an important part of the theory of stochastic differential equations. The effects of intrinsic noise within physical phenomena are ignored when using deterministic differential equations for their modelling. Stochastic differential equations (SDEs) include a random term which describes the randomness of the system as well. A general scalar SDE has the form $\mathrm{d}X(t) = F(t, X(t))\,\mathrm{d}t + G(t, X(t))\,\mathrm{d}W(t)$, where $F : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the drift coefficient and $G : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient. $W(t)$ is the so called Wiener process (see [3]), a stochastic process representing the noise. We can represent the SDE in the integral form

$$X(t) = X(0) + \int_{t_0}^{t} F(s, X(s))\,\mathrm{d}s + \int_{t_0}^{t} G(s, X(s))\,\mathrm{d}W(s),$$

where the first integral is an ordinary Riemann integral. Since the sample paths of a Wiener process do not have bounded variation on any time interval, the second integral cannot be a Riemann-Stieljtes integral. K. Itô proposed a way to overcome this difficulty with the definition of a new type of integral, a stochastic integral which is now called the Itô integral (see [3]). The solution of a stochastic differential equation is a stochastic process.

2 The Kalman - Bucy filter

The 1-dimensional Kalman-Bucy filter. Let $V(t), U(t)$ be 2 independent Wiener processes and X_0, Z_0 be random variable independent of $V(t)$ and $U(t)$. The solution $\hat{X}(t)$ of the 1-dimensional linear filtering problem of the linear system

$$\mathrm{d}X(t) = F(t)X(t)\,\mathrm{d}t + C(t)\,\mathrm{d}V(t), \quad X(0) = X_0; \quad F(t), C(t) \in \mathbb{R} \tag{1}$$

with linear observations

$$\mathrm{d}Z(t) = G(t)X(t)\,\mathrm{d}t + D(t)\,\mathrm{d}U(t), \quad Z(0) = Z_0; \quad G(t), D(t) \in \mathbb{R} \tag{2}$$

satisfies the stochastic differential equation

$$\mathrm{d}\hat{X}(t) = \left(F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right)\hat{X}(t)\,\mathrm{d}t + \frac{G(t)S(t)}{D^2(t)}\,\mathrm{d}Z(t); \qquad \hat{X}(0) = E[X_0] \tag{3}$$

where $S(t) = E[(X(t) - \hat{X}(t))^2]$ satisfies the deterministic Riccati equation

$$S'(t) = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t), \quad S(0) = E[(X_0 - E[X_0])^2]. \tag{4}$$

In other words the filtering problem is to find the best estimate $\hat{X}(s)$ of $X(s)$ satisfying (1), based on observations $Z(s)$ at times $s \le t$.

3 An application to the inductance-resistance circuit

Let $I(t)$ denote the current in an inductance-resistance circuit. Then

$$L\, I'(t) + R\, I(t) = \sigma\, \xi(t),$$

where $\xi(t)$ is a rapidly fluctuating electromagnetic force generated by the thermal noise, that we idealize as a "white noise". This electrical problem leads to the stochastic differential equation

$$\mathrm{d}I(t) = -\frac{R}{L} I(t)\, \mathrm{d}t + \frac{\sigma}{L}\, \mathrm{d}V(t), \quad I(0) = I_0. \tag{5}$$

Here $V(t)$ is the Wiener process, $\frac{R}{L}$ is the coefficient of friction and $\frac{\sigma}{L}$ is the diffusion coefficient. First we solve this problem for the circuit without any measurement. Let the initial current I_0 be a random variable with $E[I_0] < \infty$ and $V[I_0] < \infty$. To get the solution we use the Itô formula and we obtain the solution as a stochastic process

$$I(t) = e^{-\frac{R}{L}t} I_0 + \frac{\sigma}{L} \int_0^t e^{-\frac{R}{L}(t-s)}\, \mathrm{d}V(s),$$

with expectation $E[I(t)] = e^{-\frac{R}{L}t} \cdot E[I_0]$ and variance $V[I(t)] = e^{-\frac{2R}{L}t} \cdot V[I_0] + \frac{\sigma^2}{2LR}(1 - e^{-\frac{2R}{L}t})$. Let us provide some measurement of the current continuously up to time $s \le t, t > 0$. As described above, we can get from this measurement the observation equation

$$\mathrm{d}Z(t) = I(t)\, \mathrm{d}t + \mathrm{d}U(t). \tag{6}$$

Now we face to the filtering problem: To find the best estimate of the current $\hat{I}(t)$, under observations (6), while the equation (5) holds. According to the Kalman-Bucy filter, we have

$$\mathrm{d}\hat{I}(t) = \left(-\frac{R}{L} - S(t)\right) \hat{I}(t)\, \mathrm{d}t + S(t)\, \mathrm{d}Z(t); \qquad \hat{I}(0) = E[I_0] = 0 \tag{7}$$

where $S(t) = E[(I(t) - \hat{I}(t))^2]$ satisfies the deterministic Riccati equation

$$S'(t) = -\frac{2R}{L} S(t) - S^2(t) + \frac{\sigma^2}{L^2}, \quad S(0) = E[(I_0)^2] = A^2. \tag{8}$$

After we solve the Riccati equation, we get the filter in form

$$\mathrm{d}\hat{I}(t) = \left(-\frac{R}{L} - \gamma\, \frac{1 + e^{C-2\gamma t}}{1 - e^{C-2\gamma t}}\right) \hat{I}(t)\, \mathrm{d}t + \gamma\, \frac{1 + e^{C-2\gamma t}}{1 - e^{C-2\gamma t}}\, \mathrm{d}Z(t); \quad \hat{I}(0) = E[I_0] = 0, \tag{9}$$

where

$$\gamma = \frac{\sqrt{R^2 + \sigma^2}}{L}, \qquad C = \ln \left| \frac{A^2 - \frac{R}{L} - \gamma}{A^2 - \frac{R}{L} + \gamma} \right|.$$

References

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