An Application of the Kalman-Bucy filter to electrical circuits

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Abstract

This paper deals with the filtering problem, an important part of the theory of stochastic differential equations. We present an application of the continuous Kalman-Bucy filter to a special problem of RL electrical circuit.

1 Introduction

The effects of intrinsic noise within physical phenomena are ignored when using deterministic differential equations for their modelling. Stochastic differential equations (SDEs) include a random term which describes the randomness of the system as well. A general scalar SDE has the form $\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t)$, where $F : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the drift coefficient and $G : \langle 0, T\rangle \times \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient. $W(t)$ is the so called Wiener process (see [3]), a stochastic process representing the noise. We can represent the SDE in the integral form

$$X(t) = X(0) + \int_{t_0}^{t} F(s, X(s))\ \mathrm{d}s + \int_{t_0}^{t} G(s, X(s))\ \mathrm{d}W(s),$$

where the first integral is an ordinary Riemann integral. Since the sample paths of a Wiener process do not have bounded variation on any time interval, the second integral cannot be a Riemann-Stieljtes integral. K. Itô proposed a way to overcome this difficulty with the definition of a new type of integral, a stochastic integral which is now called the Itô integral (see [3]). The solution of a stochastic differential equation is a stochastic process.

Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Itô formula, contains an additional term, which, roughly speaking, is due to the fact that the square of the stochastic differential $(\mathrm{d}W(t))^2$ is equal to $\mathrm{d}t$.

The 1-dimensional Itô formula. Let the stochastic process $X(t)$ be a solution of the stochastic differential equation $\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t)$ for some suitable functions F, G (see [4], p.44). Let $g(t, x) : (0, \infty) \times \mathbf{R} \to \mathbf{R}$ be a twice continuously differentiable function. Then

$$Y(t) = g(t, X(t))$$

is a stochastic process, for which

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\ \mathrm{d}t + \frac{\partial g}{\partial x}(t, X(t))\ \mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(\ \mathrm{d}X(t))^2,$$

where $(\ \mathrm{d}X(t))^2 = (\ \mathrm{d}X(t)) \cdot (\ \mathrm{d}X(t))$ is computed according to the rules

$$\mathrm{d}t \cdot\ \mathrm{d}t =\ \mathrm{d}t \cdot\ \mathrm{d}W(t) =\ \mathrm{d}W(t) \cdot\ \mathrm{d}t = 0,\ \ \mathrm{d}W(t) \cdot\ \mathrm{d}W(t) =\ \mathrm{d}t.$$

2 The Kalman - Bucy filter

Let X_0 be a random variable and let us consider the stochastic differential equation

$$\mathrm{d}X(t) = F(t, X(t))\ \mathrm{d}t + G(t, X(t))\ \mathrm{d}W(t), \quad X(0) = X_0. \tag{1}$$

In order to improve our knowledge about the solution of this stochastic differential equation, we perform observations $Z(s)$ of $X(s)$ at times $s \leq t$. However, due to inaccuracies in our measurements we don't really measure $X(s)$ but a disturbed version of it:

$$Z(s) = X(s) + \text{"noise"}$$

So in this case there are two sources of noise, the first is already built into the equation (1), the second is coming from the error of measurement. We can derive a stochastic differential equation for the observation

$$\mathrm{d}Z(t) = A(t, X(t))\ \mathrm{d}t + B(t, X(t))\ \mathrm{d}U(t), \quad Z(0) = Z_0. \tag{2}$$

where $U(t)$ is a Wiener process independent of $W(t)$ and X_0.

Now we can state the filtering problem: What is the best estimate $\hat{X}(s)$ of $X(s)$ satisfying (1), based on observations $Z(s)$ at times $s \leq t$.

The 1-dimensional Kalman-Bucy filter. Let $V(t), U(t)$ be 2 independent Wiener processes and X_0, Z_0 be random variable independent of $V(t)$ and $U(t)$. The solution $\hat{X}(t)$ of the 1-dimensional linear filtering problem of the linear system

$$\mathrm{d}X(t) = F(t)X(t)\ \mathrm{d}t + C(t)\ \mathrm{d}V(t), \quad X(0) = X_0; \quad F(t), C(t) \in \mathbb{R} \tag{3}$$

with linear observations

$$\mathrm{d}Z(t) = G(t)X(t)\ \mathrm{d}t + D(t)\ \mathrm{d}U(t), \quad Z(0) = Z_0; \quad G(t), D(t) \in \mathbb{R} \tag{4}$$

satisfies the stochastic differential equation

$$\mathrm{d}\hat{X}(t) = \left(F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right)\hat{X}(t)\ \mathrm{d}t + \frac{G(t)S(t)}{D^2(t)}\ \mathrm{d}Z(t); \qquad \hat{X}(0) = E[X_0] \tag{5}$$

where $S(t) = E[(X(t) - \hat{X}(t))^2]$ satisfies the deterministic Riccati equation

$$S'(t) = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t), \quad S(0) = E[(X_0 - E[X_0])^2]. \tag{6}$$

For precise formulation and for the proof of the Kalman-Bucy filter see [3], page 79-95.

3 An application to the inductance-resistance circuit

Let $I(t)$ denote the current in an inductance-resistance circuit. Then

$$L\ I'(t) + R\ I(t) = \sigma\ \xi(t),$$

where $\xi(t)$ is a rapidly fluctuating electromagnetic force generated by the thermal noise, that we idealize as a "white noise". This electrical problem leads to the stochastic differential equation

$$\mathrm{d}I(t) = -\frac{R}{L}I(t)\ \mathrm{d}t + \frac{\sigma}{L}\ \mathrm{d}V(t), \quad I(0) = I_0. \tag{7}$$

Here $V(t)$ is the Wiener process, $\frac{R}{L}$ is the coefficient of friction and $\frac{\sigma}{L}$ is the diffusion coefficient. First we solve this problem for the circuit without any measurement. Let the initial current I_0 be a random variable with $E[I_0] < \infty$ and $V[I_0] < \infty$. To get the solution we use the Itô formula for the function

$$g(t,x) = e^{\frac{R}{L}t}\, x.$$

We obtain the solution as a stochastic process

$$I(t) = e^{-\frac{R}{L}t} I_0 + \frac{\sigma}{L}\int_0^t e^{-\frac{R}{L}(t-s)}\ \mathrm{d}V(s),$$

with expectation $E[I(t)] = e^{-\frac{R}{L}t}\cdot E[I_0]$ and variance $V[I(t)] = e^{-\frac{2R}{L}t}\cdot V[I_0] + \frac{\sigma^2}{2LR}(1 - e^{-\frac{2R}{L}t})$. Therefore we have $E[I(t)] \to 0$ and $V[I(t)] \to \frac{\sigma^2}{2LR}$ as $t \to \infty$.
We have shown that the distribution of $I(t)$ approaches $N(0, \frac{\sigma^2}{2LR})$ as $t \to \infty$. So for large time the solution settles down into a Gaussian distribution whose variance $\frac{\sigma^2}{2LR}$ represents a balance between the random disturbing electromagnetic force $\frac{\sigma}{L}\xi(t)$ and the force $-\frac{R}{L}I(t)$.

Let us provide some measurement of the current continuously up to time $s \leq t, t > 0$. As described above, we can get from this measurement the observation equation

$$\mathrm{d}Z(t) = I(t)\ \mathrm{d}t + \ \mathrm{d}U(t). \tag{8}$$

Now we face to the filtering problem: To find the best estimate of the current $\hat{I}(t)$, under observations (8), while the equation (7) holds. According to the Kalman-Bucy filter, we have

$$\mathrm{d}\hat{I}(t) = \left(-\frac{R}{L} - S(t)\right)\hat{I}(t)\ \mathrm{d}t + S(t)\ \mathrm{d}Z(t); \qquad \hat{I}(0) = E[I_0] = 0 \tag{9}$$

where $S(t) = E[(I(t) - \hat{I}(t))^2]$ satisfies the deterministic Riccati equation

$$S'(t) = -\frac{2R}{L}S(t) - S^2(t) + \frac{\sigma^2}{L^2}, \quad S(0) = E[(I_0)^2] = A^2. \tag{10}$$

To solve the Riccati equation, we substitute in (10)

$$S(t) = M(t) - \frac{R}{L}; \qquad S'(t) = M'(t); \qquad S^2(t) = M^2(t) - 2\frac{R}{L}M(t) + \frac{R^2}{L^2}.$$

We get a separable ordinary differential equation for the function $M(t)$

$$M'(t) = \frac{R^2+\sigma^2}{L^2} - M^2(t),$$

that we can solve. For $M(t)$ we get the implicit solution $\ln\left|\frac{M(t)-\gamma}{M(t)+\gamma}\right| = C - 2\gamma t$, where

$$\gamma = \frac{\sqrt{R^2+\sigma^2}}{L}, \qquad C = \ln\left|\frac{A^2 - \frac{R}{L} - \gamma}{A^2 - \frac{R}{L} + \gamma}\right|.$$

Now we substitute back. After some computation we get the solution of the equation (10)

$$S(t) = \gamma\,\frac{1+e^{C-2\gamma t}}{1-e^{C-2\gamma t}}, \qquad \gamma = \frac{\sqrt{R^2+\sigma^2}}{L}, \qquad C = \ln\left|\frac{A^2 - \frac{R}{L} - \gamma}{A^2 - \frac{R}{L} + \gamma}\right|.$$

We substitute this to the equation (9) and get the following stochastic differential equation for the filter

$$\mathrm{d}\hat{I}(t) = \left(-\frac{R}{L} - \gamma\,\frac{1+e^{C-2\gamma t}}{1-e^{C-2\gamma t}}\right)\hat{I}(t)\;\mathrm{d}t + \gamma\,\frac{1+e^{C-2\gamma t}}{1-e^{C-2\gamma t}}\,\mathrm{d}Z(t); \quad \hat{I}(0) = E[I_0] = 0. \tag{11}$$

For large t we have $S(t) \approx \gamma = \frac{\sqrt{R^2+\sigma^2}}{L}$ and the filtering equation becomes to the equation

$$\mathrm{d}\hat{I}(t) = -\frac{R+\sqrt{R^2+\sigma^2}}{L}\;\hat{I}(t)\;\mathrm{d}t\; + \frac{\sqrt{R^2+\sigma^2}}{L}\;\mathrm{d}Z(t). \tag{12}$$

This equation can be solved using the Itô formula for the function $g(t,x) = e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,t}\;x$. We have

$$\mathrm{d}\left(e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,t}\hat{I}(t)\right) = \frac{R+\sqrt{R^2+\sigma^2}}{L}\;e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,t}\;\hat{I}(t)\;\mathrm{d}t + e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,t}\;\mathrm{d}\hat{I}(t) =$$

$$= \frac{\sqrt{R^2+\sigma^2}}{L}\;e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,t}\;\;\mathrm{d}Z(t).$$

Thus

$$\hat{I}(t) = \frac{\sqrt{R^2+\sigma^2}}{L}\int_0^t e^{\frac{R+\sqrt{R^2+\sigma^2}}{L}\,(s-t)}\;\;\mathrm{d}Z(t) \tag{13}$$

is the solution of the filtering problem for the RL circuit.

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