Two examples of the constructions of non-continuous t-norms

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Abstract

In this contribution we present some interesting constructions of non-continuous triangular norms.

The first approach is based on strictly increasing sequences of natural numbers. An associative commutative monotone and bounded by minimum binary operation on these sequences induces a t-norm. The corresponding t-norm is left continuous and therefore it is applicable in the fuzzy logic. Several other interesting properties of this t-norm are investigated, including its residual implicator.

Second approach uses idea of multiplicative generator φ of a triangular norm is a special monotone function $\varphi : [0,1] \to [0,1]$ with fixed point 1 and $\varphi(0) < 1$. The corresponding t-norm T is defined by means of φ as follows:

$$T^*(x,y) = \varphi^{(-1)}(\varphi(x).\varphi(y)),$$

where $\varphi^{(-1)} : [0,1] \to [0,1]$ is a so-called pseudo-inverse of φ. If strictly increasing function φ is left continuous, but non-continuous, then associativity of induces operator is violated, [6]. However, then the operation

$$T_*(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \varphi(\varphi^{(-1)}(x).\varphi^{(-1)}(y)) & \text{otherwise,} \end{cases}$$

defines a non-continuous t-norm. Some example will be given.

1 Introduction

First we can recall some important definitions and properties of triangular norms.

Definition 1 *A triangular norm (t-norm for short) is a binary operation on the unit interval* $[0,1]$, *i.e., a function* $T : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$ *the following four axioms are satisfied:*

- *(T1) Commutativity*
$$T(x,y) = T(y,x),$$
- *(T2) Associativity*
$$T(x,T(y,z)) = T(T(x,y),z),$$
- *(T3) Monotonicity*
$$T(x,y) \le T(x,z) \quad \textit{whenever} \;\; y \le z,$$
- *(T4) Boundary Condition*
$$T(x,1) = x.$$

Proposition 1 *A t-norm T is left-continuous if and only if it is left-continuous in its first component, i.e., if for each $y \in [0,1]$ and for each non - decreasing sequence $(x_n)_{n\in N} \in [0,1]^N$ we have*

$$\sup_{n\in N} T(x_n, y) = T(\sup_{n\in N} x_n, y).$$

Now, we will turn our attention to the Archimedean property. We recall another definition of Archimedean t-norms, which is equivalent with the classical one.

Proposition 2 *A t-norm T is Archimedean if and only if for each $x \in]0,1[$ we have*

$$\lim_{n\to\infty} \underset{n\text{-times}}{T(x, ..., x)} = 0.$$

Following important algebraic property is *strict monotonicity.*

Proposition 3 *A t-norm T is strictly monotone if and only if the cancelation law holds, i.e., if $T(x,y) = T(x,z)$ and $x > 0$ imply $y = z$.*

Definition 2 *Let T be a t-norm. An element $a \in]0,1[$ is called a nilpotent element of T if there exists some $n \in N$ such that $a_T^{(n)} = 0$.*

Multiplicative generator φ of a triangular norm is a strictly increasing function $\varphi : [0,1] \to [0,1]$ such that $\varphi(1) = 1$ and $\varphi(x).\varphi(y) \in H(\varphi)$ or $\varphi(x).\varphi(y) < \varphi(0)$. The corresponding t-norm T is defined by means of φ as follows:

$$T(x,y) = \varphi^{(-1)}(\varphi(x).\varphi(y)),$$

where $\varphi^{(-1)} : [0,1] \to [0,1]$ is a so-called pseudo-inverse of φ defined by

$$\varphi^{(-1)}(t) = \sup(x \in [0,1]; \varphi(x) < t)$$

with convention $\sup \emptyset = 0$.

Continuous t-norms which are not Archimedean cannot be generated by means of multiplicative (additive) generator. However, there are several non-continuous t-norms which are generated [5], e.g. the drastic product T_W. Assuming the left-continuity of a t-norm, note that only generated t-norms are then continuous and consequently Archimedean. On the other hand, there are examples of non-continuous generated t-norms which are non-Archimedean (and then necessarily not continuous).

2 The construction based on sequences

In theory of t-norms we know some standard constructions, for example ordinal sums, $\varphi-$transformations and constructions with additive or multiplicative generators. In this contribution we will deal with non-standard types of constructions of t-norms. The proofs of following propositions we can find in [7] and [8]. For $x \in]0,1]$, we can write

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}},$$

which is the unique infinite dyadic expansion of x, where $(x_i)_{i\in N}$ is strictly increasing sequence of natural numbers. It is easy to see that each $x \in]0,1]$ is in a one to one correspondence with $(x_i)_{i\in N}$ strictly increasing sequence of natural numbers.

$$x \approx (x_i)_{i\in N}$$

Remark 1 *Let* $x \approx (x_i)_{i \in N}$ *and* $y \approx (y_i)_{i \in N}$. *Then* $x < y$ *if and only if there exists* $k \in N$ *such that for all* $i \in N$, $i < k$, *we have* $x_i = y_i$ *and* $x_k > y_k$.

Some of recently introduced new t-norms based on above described dyadic expansion are recalled in the following example. These t-norms are not continuous and have a dense set of discontinuity points, see Budinčevič and Kuriič [1].

Example 1 *For* $(x, y) \in \ ]0, 1]^2$ *let*

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}} \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{1}{2^{y_i}},$$

be the unique dyadic representations of x *and* y. *Then the t-norm* $T_1 : [0, 1]^2 \to [0, 1]$ *is given by*

$$T_1(x, y) = \begin{cases} \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i + y_i}}, & \text{if } (x, y) \in]0, 1[^2, \\ \min(x, y), & \text{otherwise,} \end{cases}$$

and the t-norm $T_2 : [0, 1]^2 \to [0, 1]$ *is given by*

$$T_2(x, y) = \begin{cases} \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i \cdot y_i}}, & \text{if } (x, y) \in]0, 1[^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Both T_1 *and* T_2 *are Archimedean and strictly monotone t-norms, which are neither left nor right continuous.*

Based on the original idea of Budinčevič and Kurilič [1], the reals from the half-open interval]0,1] are transformed into the strictly increasing sequences of natural numbers. An associative commutative monotone (and bounded by minimum) binary operation on strictly increasing sequences of natural numbers induces a t-norm. However, the mentioned binary operation need not be the coordinatewise extension of some given binary operation on natural numbers as proposed in [1]. The usual requirement in fuzzy logic to a t-norm T to model a conjuction is its left-continuity. Then, the implication can be modeled by the corresponding residual operator. Therefore we will investigate t-norms with similar properties as in above example under additional requirement of their left-continuity.

Proposition 4 *For* $(x, y) \in \ ]0, 1]^2$ *let*

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}} \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{1}{2^{y_i}},$$

$$x \approx (x_i)_{i \in N} \quad \text{and} \quad y \approx (y_i)_{i \in N}$$

be the unique dyadic representations of x *and* y. *Let* $T_* : [0, 1]^2 \to [0, 1]$ *be given by*

$$T_*(x, y) = \begin{cases} 0, & \text{if } \min(x, y) = 0, \\ \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i + y_i - i}}, & \text{otherwise.} \end{cases}$$

Then T_* *is a strictly monotone t-norm.*

Remark 2 *Note, that T_* is left continuous. Each point $(x,y) \in \ ]0,1[^2$, where at least one coordinate has a finite dyadic representation, is a discontinuity point. We can see the set of all discontinuity points of t-norm T_* is dense in the unit square. Recall that the left-continuity of T_* allows its application in the framework of fuzzy logic to model a non-continuous fuzzy conjunction. This result is also a negative answer to the open problem of E.Pap (Is a strictly monotone t-norm continuous in point $(1,1)$ necessarily always continuous ?), see [3]. Now, we will turn our attention to the Archimedean property. Indeed, for any $x \in]1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]; n \in N$;*

$$\lim_{m\to\infty} \underset{m\text{-times}}{T_*(x,...,x)} = 1 - \frac{1}{2^n},$$

violating the Archimedean property of T_*. Now, we stress that T_* is an example of a strictly monotone t-norm, which is not Archimedean.

Let T be a left-continuous t-norm which models a fuzzy conjuction. A standard way how to introduce a fuzzy implication based on T uses the residual operator R_T

$$R_T(x,y) = sup(z \in [0,1]; T(x,z) \le y).$$

For t-norm T_* we obtain the following residuation:

$$R_{T_*}(x,y) = \begin{cases} \min(1, \frac{y}{x}), & \text{if } x \le y \text{ or } x = \frac{1}{2^n}, \\ \sum\limits_{i=1}^{k} \frac{1}{2^{y_i - x_i + i}} + \frac{1}{2^{y_k - x_k + k}}, & \text{otherwise,} \end{cases}$$

where $(y_i - x_i + i)_{i\in N}$ is an increasing sequence for $i \in \{1,...,k\}$ and $y_k - x_k + k \ge y_{k+1} - x_{k+1} + k + 1$. Note that $\frac{0}{0} = 1$ by convention.

Another t-norm with similar properties as T_* is T_{**},

$$T_{**}(x,y) = \begin{cases} 0, & \text{if } min(x,y) = 0, \\ \sum\limits_{i=1}^{\infty} \frac{1}{2^{(x_i - i+1).(y_i - i+1)+i-1}}, & \text{otherwise.} \end{cases}$$

Finally, a family of t-norms $(T_k)_{k\in N}$ with similar properties is given by:

$$T_k(x,y) = \sum_{i=1}^{k} \frac{1}{2^{x_i + y_i - i}} + \sum_{i=k+1}^{\infty} \frac{1}{2^{(x_i - i+1).(y_i - i+1)+i-1}}.$$

Remark 3 *Note that t-norms T_* and T_{**} can be understood as limit members of the family $(T_k)_{k\in N}$*

$$T_* = T_\infty \quad and \quad T_{**} = T_0.$$

3 The construction based on idea of multiplicative generators

For $x \in]0,1]$, we can write

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i},$$

which is the unique infinite dyadic expansion of x, where $x_i \in \{0,1\}$ for $i \in N$. The set $\{i; x_i = 1\}$ is infinite. It is easy to see that each $x \in \ ]0,1]$ is in a one to one correspondence with $(x_i)_{i\in N}$, where $x_i \in \{0,1\}$ and card $\{i; x_i = 1\}$ is infinite. We will use the following notation:

$$x \approx (x_i)_{i\in N}.$$

Remark 4 *Let $x \approx (x_i)_{i\in N}$ and $y \approx (y_i)_{i\in N}$. Then $x < y$ if and only if there exists $k \in N$ such that for all $i \in N$, $i \le k$, we have $\quad x_i = y_i \quad$ and $\quad x_{k+1} < y_{k+1}$.*

We discuss a t-norm based on above described dyadic expansion, which is generated by non-continuous multiplicative generator.

Proposition 5 *Let function $g : [0,1] \to [0,1]$ be given by*

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sum\limits_{i=1}^{\infty} \frac{2.x_i}{3^i} & \text{otherwise,} \end{cases}$$

where if $x \in]0,1]$ then $x \approx (x_i)_{i\in N}$. Function g is a strictly increasing and left-continuous. Each finite dyadic rational is point of discontinuity of function g.

Remark 5 *We can define $f(x) = g^{(-1)}(x) = \sup(z \in [0,1]; g(z) < x)$ as pseudo-inverse of function g. Because of properties of this function, new function f is continuous and $f^{(-1)} = g$. Note that function f is well-known in function theory as the Cantor function.*

Example 2 *Let $T^S : [0,1]^2 \to [0,1]$ be given by*

$$T^S(x,y) = f(g(x).g(y)),$$

where g is a function from Proposition 5 and f is pseudo-inverse of function g. Then T^S is operator, which is not a t-norm. T^S is commutative, monotone, it fulfiles boundary condition, however, the associativity is violated. For example

$$T^S(\frac{1}{2}, T^S(\frac{3}{4}, \frac{3}{4})) = \frac{1}{4} < \frac{1}{2} = T^S(T^S(\frac{1}{2}, \frac{3}{4}), \frac{3}{4}).$$

On the other hand, the operation

$$T_S(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ g(f(x).f(y)) & \text{otherwise,} \end{cases}$$

defines a t-norm. The axioms $T(1), T(3), T(4)$ are evidently fulfilled. Concerning the associativity, it follows from the continuity of f.

Remark 6 *Note that for given functions g and f we have $f(g(x)) = x$ for $x \in]0,1]$. Therefore $T_S(x, T_S(x,x)) = g(f(x)^3)$ for $x \in]0,1[$, which implies $x_{T_S}^{(n)} = g(f(x)^n)$ for $x \in\ ]0,1[$.*

Remark 7 *Now, we will turn our attention to important properties of this t-norm.*

- *The t-norm T_S is left-continuous on $[0,1[^2$.*
- *The t-norm T_S is continuous in point $(1,1)$.*
- *The t-norm T_S is not left-continuous on $[0,1]^2$.*
- *The t-norm T_S is Archimedean.*
- *The t-norm T_S is not strictly monotone.*

The t-norm T_S is an example of t-norm which is left-continuous on $[0,1[^2$ and continuous in point $(1,1)$, but non-left-continuous on $[0,1]^2$. More, this t-norm is Archimedean but neither strictly monotone nor nilpotent in any point from $]0,1[$.

Indeed, let $a \in]\frac{1}{3^n}, \frac{2}{3^n}]$ for some $n \in N$. Then $a_{T_S}^{(m)} = \frac{1}{3^{m.n}} > 0$ for all $m \in N$. Consequently, no element $b \in]0,1[$, $b > a$, can be a nilpotent element of T_S, showing the non-existence of any nilpotent element of T_S.

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