Binary trees and power series

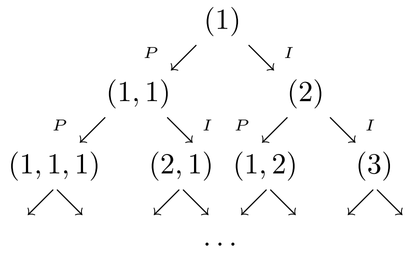
Petr Fuchs
Technical University of Brno,
Faculty of Electrical Engineering and Communication, Department of Mathematics
e-mail: fuchsp@feec.vutbr.cz

Let Ω be the set of all m-tuples $(k_1, \ldots, k_m)$ with $m, k_1, \ldots, k_m$ positive integers. We define operators $P, I : \Omega \to \Omega$ as follows,

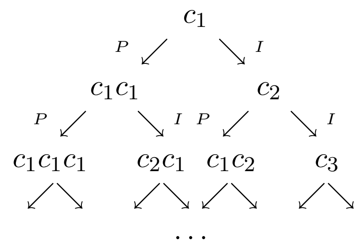
$$\begin{aligned} P(k_1, k_2, \ldots, k_m) &= (1, k_1, k_2, \ldots, k_m), \\ I(k_1, k_2, \ldots, k_m) &= (k_1 + 1, k_2, \ldots, k_m). \end{aligned}$$

Operator P makes an $(m+1)$-tuple by putting 1 at the beginning and operator I increases by 1 the first number in the given m-tuple.

We now consider a binary tree whose nodes are m-tuples $(k_1, \ldots, k_m) \in \Omega$, the initial node being (1), and the two children of a given node $(k_1, \ldots, k_m)$ are $P(k_1, \ldots, k_m)$ and $I(k_1, \ldots, k_m)$.



Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers. In the above tree, we will replace the nodes $(k_1, \ldots, k_m)$ with products $c_{k_1} \cdots c_{k_m}$. We obtain a tree that will be called the PI-tree of the sequence $(c_n)_{n=1}^{\infty}$.



We now ask the following question: in the above PI-tree of the sequence $(c_n)_{n=1}^{\infty}$, what is the sum of the nodes at the nth level (counting the initial node as the first level)? The answer to this question is posible in terms of power series.

Suppose that the function

$$g(t) = \sum_{n=1}^{\infty} c_n t^n$$

is analytical at $t = 0$. Put

$$f(t) = \frac{g(t)}{1 - g(t)}$$

and expand it into a power series,

$$f(t) = \sum_{n=1}^{\infty} x_n t^n.$$

Then x_n is the sum of the nodes at the nth level of the PI-tree of the sequence $(c_n)_{n=1}^{\infty}$.

The above result may be demonstrated on the sequence

$$c_n = -(n+1).$$

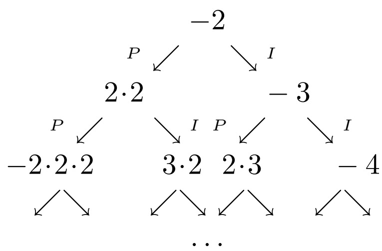
The PI-tree of this sequence has nodes

$$c_{k_1} \cdots c_{k_m} = (-1)^m (k_1+1) \cdots (k_m+1),$$

each node has two children,

$$\begin{aligned} P\Big((-1)^m\,(k_1+1)\cdots(k_m+1)\Big) &= (-1)^{m+1}\,2\,(k_1+1)\cdots(k_m+1),\\ I\Big((-1)^m\,(k_1+1)\cdots(k_m+1)\Big) &= (-1)^m\,(k_1+2)\,(k_2+1)\cdots(k_m+1), \end{aligned}$$

and the initial node is $c_1 = -2$.



The sum of the nodes at the nth level is, for $n \geq 3$, $x_n = 0$. It follows from

$$g(t) = \sum_{n=1}^{\infty} c_n t^n = \sum_{n=1}^{\infty} (-n-1) t^n = 1 - \frac{1}{(t-1)^2}$$

and

$$f(t) = \frac{g(t)}{1-g(t)} = -2t + t^2 = \sum_{n=1}^{\infty} x_n t^n.$$

References

[1] Petr Fuchs. Bernoulli numbers and binary trees. *Tatra Mt. Math. Publ*, **20**: 111–117, 2000.