On two-scale convergence

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Abstract
Two-scale convergence is an important tool in homogenization theory. The contribution deals with its alternative definition based on two-scale mapping and transform. It removes the problems with choice of the space of admissible test functions and simplifies the proofs.

1 Introduction

Two-scale convergence introduced by Nguetseng and Allaire is an important tool in homogenization theory. It enables to overcome problem of passing to the limit in product of two weakly converging sequences: e.g. sequences $u^\varepsilon(x) = v^\varepsilon(x) = \sin(x/\varepsilon)$ converge to zero weakly in $L^2(0,\pi)$, but $u^\varepsilon v^\varepsilon$ converge to a nonzero function 1/2.

In homogenization theory we need to pass to the limit in product of the periodic coefficients and weakly converging sequence of the solutions. The problem was solved by a special choice of a sequence of test functions by A. Bensoussans, J. L. Lions and G. Papanicolaou, later by the div-curl lemma of F. Murat and L. Tartar. Simpler solution appeared in two-scale convergence.

In the contribution we discuss problems of classical definition of the two-scale convergence, and propose an alternative definition which simplifies definition and proofs.

Notation. All sequences will be denoted by superscript ε. The sequence $\{\varepsilon_k\}$ of small positive reals $\varepsilon_k \to 0$ is called scale, but the subscript k is usually omitted and sequences written u^ε. Symbol Y means a basic cell usually the cube $Y = \langle 0,1)^N$. We shall say that a function $a(y)$ is Y-periodic, if it is defined on $\mathbb{R}^N$ and is 1-periodic in each variable y_i. Spaces of Y-periodic functions are denoted by $X(Y_\#)$. Its elements are Y-periodic in $\mathbb{R}^N$ and their restriction to any bounded domain $G \subset \mathbb{R}^N$ is in $X(G)$, although the norm is taken over the cell Y only.

2 Classical definition of the two-scale convergence

Definition 1 *A sequence $u^\varepsilon(x)$ is said to two-scale converge to a limit $u^0(x,y) \in L^2(\Omega \times Y)$ iff*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u^0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{1}$$

for all admissible $\varphi(x,y)$ from a space X of functions Y-periodic in y.
If, in addition, $\lim \|u^\varepsilon\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega\times Y)}$ u^ε is said to converge strongly two-scale.

The space $X \subset L^2(\Omega \times Y)$, usually $X = C(\Omega, L^2(Y_\#))$ is used.

The choice of the space X is important, it cannot be neither too large, nor too small. We cannot choose $X = L^2(\Omega \times Y)$ since on the left hand side of (1) the test function $\varphi(x, x/\varepsilon)$ takes values on a zero measure subset of $\Omega \times Y$, which for integrable function $\varphi(x,y)$ is not defined. Thus in the classical definition some continuity of test functions $\varphi(x,y)$ must be assumed.

To overcome all the problems of choice of the space X we introduce an alternative approach based on two-scale mapping and transform called also periodic unfolding. The basic ideas appeared in [1] [2], [3], see also [4].

3 Alternative approach

For a given scale $\{\varepsilon\}$ the two-scale transform converts each function $u^\varepsilon(x)$ of x-variables to a function $\widehat{u^\varepsilon}(x, y)$ of both x and y variables. Then the convergence is tested in $L^2(\Omega \times Y)$. In this approach both the limit and test functions can be taken from the maximal space $X = L^p(\Omega \times Y_\#)$ and we need not care of the space X. This approach also simplifies the proofs. It enables to introduce also the strong two-scale convergence in a natural way.

Two-scale mapping and transform. We define a measure preserving mapping τ^ε which maps $\Omega \times Y$ onto Ω. The system of k-shifted and ε-scaled cells $Y_k^\varepsilon = \varepsilon(Y + k)$ for $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ covers the whole space $\mathbb{R}^n$. We take a subsystem of these disjoint cells covering the domain $\Omega \subset \mathbb{R}^N$. It consists of inner complete cells $Y_k^\varepsilon \subset \Omega$ and boundary uncomplete cells $\widetilde{Y}_k^\varepsilon = Y_k^\varepsilon \cap \Omega$.

Definition 2 *Two-scale mapping $\tau^\varepsilon : \Omega \times Y \to \Omega$ in the previous notation is defined by*

$$\tau^\varepsilon(x, y) = \begin{cases} \varepsilon k + \varepsilon y & \text{if } x \in Y_k^\varepsilon \subset \Omega \quad \text{(inner cells)} \\ x & \text{if } x \in \widetilde{Y}^\varepsilon \quad \text{(boundary cells)} \end{cases} \tag{2}$$

and two-scale-transform $T^\varepsilon : X(\Omega) \to X(\Omega \times Y)$ is defined by

$$T^\varepsilon : u^\varepsilon(x) \mapsto \widehat{u^\varepsilon}(x, y) \equiv (T^\varepsilon u^\varepsilon)(x, y) = u^\varepsilon(\tau^\varepsilon(x, y)) \,. \tag{3}$$

Definition 3 *Let $\{\varepsilon\}$ be a scale and $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$.*

We say that a sequence $\{u^\varepsilon\}$ in $L^p(\Omega)$ two-scale converge (strongly two-scale converge, resp.) with respect to the scale $\{\varepsilon\}$ in $L^p(\Omega)$ to the limit $u_0(x, y) \in L^p(\Omega \times Y)$ iff

$$\widehat{u^\varepsilon} = (T^\varepsilon u^\varepsilon) \quad \text{converge to} \quad u_0 \quad \text{in} \quad L^p(\Omega \times Y) \quad \text{weakly (strongly, resp.).}$$

This approach yields the following results directly from the L^p theory:

Theorem 4 (Compactness) *Let $\{\varepsilon\}$ be a scale and $\{u^\varepsilon\}$ a bounded sequence in $L^p(\Omega)$. Then there exists a subscale $\{\varepsilon'\} \subset \{\varepsilon\}$ and a limit $u^* \in L^p(\Omega \times Y)$ such that $u^{\varepsilon'}$ (weakly) two-scale converge to u^* with respect to the subscale $\{\varepsilon'\}$ (in case $p = \infty$ weakly* two-scale converge).*

Theorem 5 (Main convergence result) *Let a sequence u^ε two-scale (weakly) converge to u_0 and sequence v^ε strongly two-scale converges to v^ε, both with respect to the same scale $\{\varepsilon\}$, the former in $L^p(\Omega)$ and the latter in $L^q(\Omega)$, assuming $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$.*

Then the product $u^\varepsilon v^\varepsilon$ two-scale converges to the limit $u_0 \, v_0 \equiv u_0(x, y) \, v_0(x, y)$ in $L^r(\Omega)$.

4 Conclusion

The two-scale convergence is a refinement of the weak convergence in L^p spaces. It conserves compactness property for bounded sequences and in many cases enables to pass to the limits in product of two weakly converging sequences, see Theorem 5, particularly in the homogenization problems where the periodic coefficients strongly two-scale converge. The alternative approach in Definition 3 removes the restriction on the space of test functions and simplifies the proofs.

References

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