On Two-Scale Convergence

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Abstract

Two-scale convergence is an important tool in homogenization theory. The contribution deals with its alternative definition based on two-scale mapping and transform. It removes the problems with choice of the space of admissible test functions and simplifies the proofs.

1 Introduction

Two-scale convergence was introduced by Nguetseng [10] and Allaire [1]. It is an important tool particularly in homogenization theory. It enables to overcome the following problem:

Let u^ε and v^ε be two sequences weakly converging in $L^2(\Omega)$.
What is the limit of the their product $u^\varepsilon v^\varepsilon$, or what is the limit of $\int_\Omega u^\varepsilon v^\varepsilon \, \mathrm{d}x$?

If not more then one sequence converges weakly, then $\lim u^\varepsilon v^\varepsilon = \lim u^\varepsilon \lim v^\varepsilon$. If both sequences converge only weakly, then we cannot pass to the limit since, the corresponding weak limits do not conserve enough information on the local behavior of the functions $u^\varepsilon, v^\varepsilon$, as the following counterexample shows:

Sequences $u^\varepsilon(x) = v^\varepsilon(x) = \sin\frac{x}{\varepsilon}$ $(\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ converges weakly to zero functions, but the limits $\lim(u^\varepsilon v^\varepsilon) = \frac{1}{2}$ while $\lim u^\varepsilon \lim v^\varepsilon = 0 \cdot 0 = 0$.

This problem occurs in homogenization theory. It studies behavior of sequence of solutions to boundary value problems for equations of type

$$-\mathrm{div}\,(a^\varepsilon \, \nabla u^\varepsilon) = f$$

with periodic coefficients a^ε when period ε tends to zero. Indeed, in the weak formulation the coefficients a^ε form a weakly converging sequence. Since sequence of the solutions u^ε is bounded, it contains a weakly converging subsequence $u^{\varepsilon'}$ and we need to pass to the limit of the product $a^{\varepsilon'} \nabla u^{\varepsilon'}$ of two weakly converging sequences, which leads to the mentioned problem.

The problem was first solved by A. Bensoussans, J. L. Lions and G. Papanicolaou, see [3], with a special choice of a sequence of weakly converging periodic test functions satisfying an auxiliary adjoint problem, later by the div-curl lemma of F. Murat and L. Tartar [8]. Simpler solution appeared in two-scale convergence. Its limit $u^0(x, y)$ is a function of variable x and y, the local behavior of u^ε is conserved in the second variable y.

The aim of this paper is to introduce an alternative approach to the two-scale convergence which is more natural and simplifies the definition and proofs. In the contribution we mention the classical definition of the two-scale convergence, discuss its problems and introduce the alternative definition including its properties and some examples.

2 Preliminaries

We start with formulation of notions concerning periodic two-scale convergence which are implicitly assumed but usually not specified.

2.1 Scale

In this contribution all sequences will be denoted by a superscript ε. It is a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of small positive parameters ε_k tending to zero as $k \to \infty$. The sequence will be called the scale. This notation comes from the periodic homogenization approach, where it denotes diminishing period ε of the coefficients in the equation. In the non-periodic homogenization it is just a label.

Instead of $k \to \infty$ the sequences are denoted by $\varepsilon_k \to 0$. The subscript k is usually omitted and thus sequences are denoted by the superscript ε only, e.g. $\{u^\varepsilon\}$. Although in the definitions of two-scale convergence it is usually missing, the scale i.e. a fixed sequence of small parameters $\varepsilon_k \to 0$ is always supposed. Speaking about a converging subsequence $\{u^{\varepsilon'}\}$ the two-scale convergence is taken also with respect to its subscale $\{\varepsilon'\}$.

2.2 Domain Ω, period Y and Y-periodic functions

In this contribution Ω will denote a bounded domain in $\mathbb{R}^N$ with points $x = (x_1, \ldots, x_N)$ and a "good" boundary, e.g. Lipschitz boundary. In the periodic homogenization the symbol Y stands for a basic period in $\mathbb{R}^N$ called unit cell with points $y = (y_1, \ldots, y_N)$. It is usually a rectangular parallelepiped $Y = \langle 0, \overline{y}_1) \times \cdots \times \langle 0, \overline{y}_N)$, where $\overline{y}_i$ are positive numbers, usually satisfying $\overline{y}_1 \cdots \overline{y}_N = 1$ to obtain unit volume of the cell Y. For the sake of simplicity we shall deal with the most often case $\overline{y}_i = 1$, i.e. the period is the N-dimensional cube $Y = \langle 0,1)^N = \langle 0,1) \times \cdots \times \langle 0,1)$.

We shall say that a function $a(y)$ is Y-periodic, if it is defined on $\mathbb{R}^N$ and is periodic in each variable y_i with period $\overline{y}_i$, i.e.

$$a(y_1 + k_1, \ldots, y_N + k_N) = a(y_1, \ldots, y_N)$$

holds for each $y \equiv (y_1, \ldots, y_N) \in \mathbb{R}^N$ and $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$. If the function depends even on another variable, say x, we say that it is Y-periodic in y.

Let us recall that taking a bounded measurable Y-periodic function $a(Y)$ and a scale $\{\varepsilon\}$, $\varepsilon \to 0$, the relation

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) \equiv a\left(\frac{x_1}{\varepsilon}, \ldots, \frac{x_N}{\varepsilon}\right)$$

defines a sequence of periodic functions with diminishing period. It converges in any $L^p(\Omega)$ weakly for $p < \infty$ and weakly-* for $p = \infty$ to a constant function $\overline{a}$, being the integral average of $a(y)$, i.e. $\overline{a} = \int_Y a(y)\,\mathrm{d}y$.

2.3 Spaces of periodic functions

Besides of the Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{k,p}(\Omega)$ and spaces $C^k(\overline{\Omega})$, we shall need spaces of Y-periodic functions denoted by $X(Y_\#)$. Its elements $a(y)$ are defined on $\mathbb{R}^n$, they are Y-periodic in y and their restriction to any bounded domain $G \subset \mathbb{R}^N$ is in $X(G)$, although the norm is taken over the cell Y only. Thus $X(Y_\#)$ can be smaller than the space $X(Y)$ extended to $\mathbb{R}^N$ by periodicity. While $L^2(Y_\#)$ can be identified with $L^2(Y)$, the functions of $C^k(Y_\#)$ or $W^{1,p}(Y_\#)$ have, in addition, equal values on the opposite sides of the cell Y.

Further spaces of abstract functions will be used. The following spaces $L^p(\Omega, L^p(Y_\#))$, $L^p(Y_\#, L^p(\Omega))$ and $L^p(\Omega \times Y_\#)$ can be identified, since they have the same norms and smooth Y-periodic function are dense in each space.

3 The classical two-scale convergence

DEFINITION 1 *A sequence $u^\varepsilon(x)$ is said to two-scale converge to a limit $u^0(x,y) \in L^2(\Omega \times Y)$ iff*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u^0(x,y)\varphi(x,y)\, \mathrm{d}y\, \mathrm{d}x \tag{1}$$

for all admissible $\varphi(x,y)$ from a space X of functions Y-periodic in y and $X \subset L^2(\Omega \times Y)$. If, in addition, $\lim \|u^\varepsilon\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega\times Y)}$ we say that u^ε converge strongly two-scale.

Usually $X = C(\Omega, L^2(Y_\#)$ is taken.

3.1 The problem of X – the space for test functions

The choice of the space X is important, it cannot be neither too large, nor too small. We cannot choose $X = L^2(\Omega \times Y)$ since in the limit (1) the test function $\varphi(x,y)$ is transformed into one variable function by $\varphi(x, x/\varepsilon)$, which for integrable function $\varphi(x,y)$ is not defined. Indeed, taking into account periodicity of φ, for each ε in $\varphi(x, x/\varepsilon)$ the values of $\varphi(x,y)$ on finite or countable paralel segments in $\Omega \times Y$ are used.

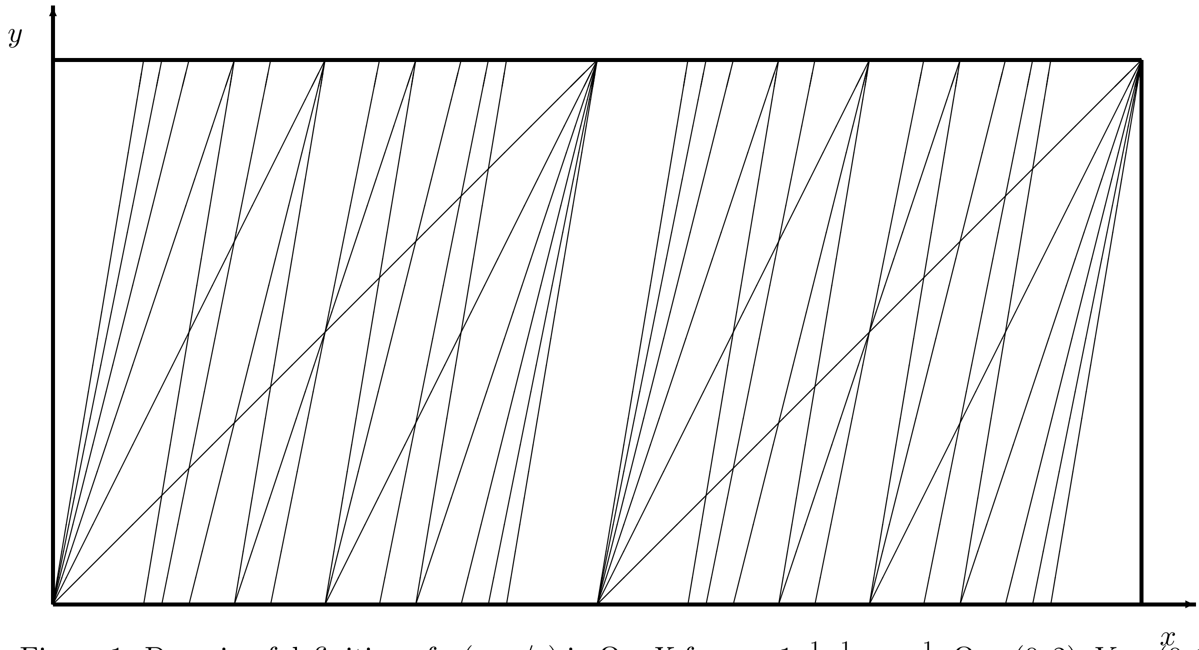

Figure 1: Domain of definition of $\varphi(x, x/\varepsilon)$ in $\Omega \times X$ for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{6}$, $\Omega = (0,2)$, $Y = (0,1)$.

Even if the countable sequence of periods ε_k is taken, it is still a zero measure set in $\Omega \times Y$ and elements of $L^2(\Omega \times Y)$ are classes of functions that may differ on zero measure subsets. Thus in the classical definition some continuity of the test functions $\varphi(x,y)$ must be assumed.

On the other hand X cannot be too small. In case $X = C_0^\infty(\Omega, C^\infty(Y_\#))$ of the first papers the definition admits even sequences unbounded in $L^2(\Omega)$, e.g., see [11], the sequence $u^\varepsilon(x) = \frac{1}{\varepsilon}$ for $x \in (0, \varepsilon)$, otherwise $u^\varepsilon(x) = 0$ satisfies the convergence (1) for any test function with compact support in $(0,1)$, but it is unbounded in $L^2(0,1)$ and thus cannot converge weakly in $L^2(0,1)$. Thus for "small" spaces X the definition (1) is completed by the requirement that the sequence u^ε is bounded in $L^2(\Omega)$.

In the applications we need the space X as small as possible to simplify verification of the two-scale convergence and big enough to admit passing to the limit for as much functions as possible.

4 Alternative approach

To overcome all problems of choice of the space X we introduce an alternative approach based on the so-called two-scale mapping and two-scale transform. The basic ideas appeared in [2], [4], see also [9]. For a given scale $\{\varepsilon\}$ the two-scale transform converts each function $u^\varepsilon(x)$ of x-variables into a function $\widehat{u^\varepsilon}(x,y)$ of both x and y variables. Then the convergence is tested in space $L^2(\Omega \times Y)$.

Using this approach both the limit and test functions can be taken from the maximal space $X = L^p(\Omega \times Y_\#)$ and we need not take care of the space X. This approach also simplifies the proofs. It enables to introduce also the strong two-scale convergence in a natural way. In addition compactness and passing to the limit in homogenization theory also follows directly from the theory of L^p-spaces.

First, the transform will be described for 1D case.

4.1 Two-scale mapping and transform – 1D case

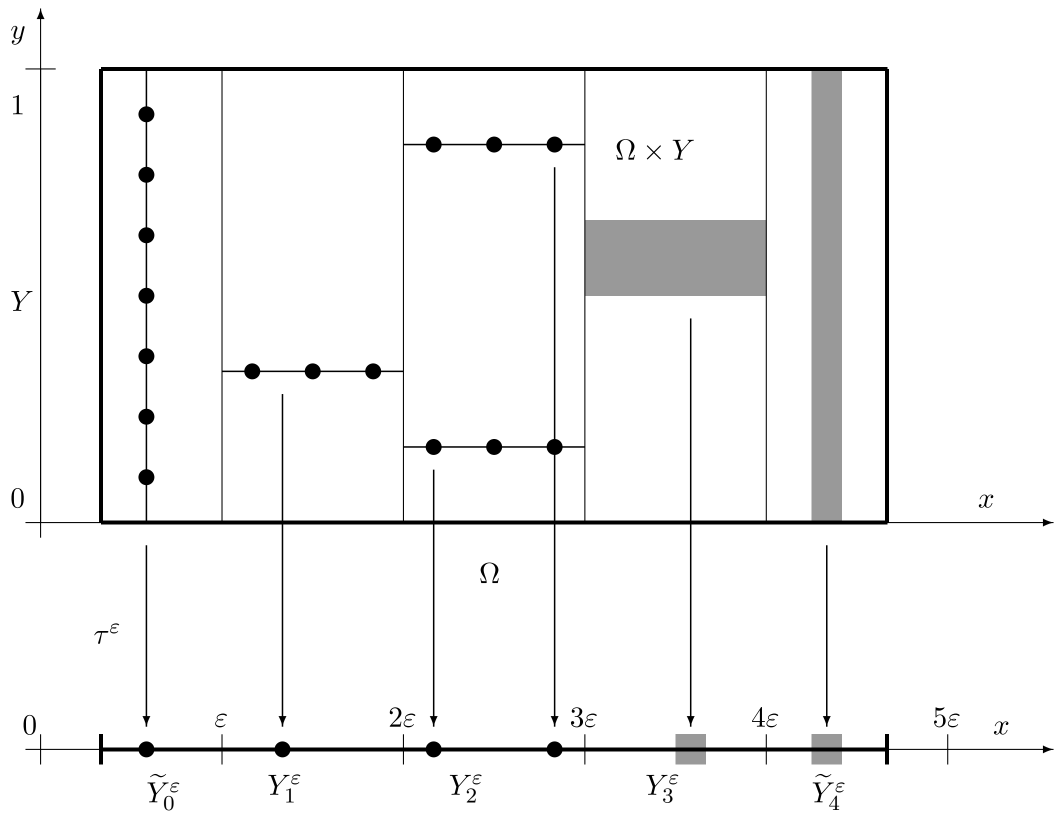

Figure 2: Two-scale mapping: $\tau^\varepsilon : \Omega \times Y \to \Omega$

We define a mapping τ^ε dependent on ε which maps $\Omega \times Y$ onto Ω. Let $\varepsilon > 0$, $Y = \langle 0, 1)$ and $\Omega = (\omega_l, \omega_r)$. Let the multiples $\varepsilon \cdot k$ ($k \in \mathbb{Z}$, $k = k_l, k_l + 1, \ldots, k_r$) be the inner points of the interval $\Omega = (\omega_l, \omega_r)$. They decompose it into a finite number of "complete" inner cells $Y_k^\varepsilon = \varepsilon Y + \varepsilon k = \langle k\varepsilon, (k+1)\varepsilon)$ for $k = k_l, k_l+1, \ldots, k_r - 1$ and at most two uncomplete boundary cells $\widetilde{Y}_{k_l-1}^\varepsilon = Y_{k_l-1}^\varepsilon \cap \Omega = (\omega_l, k_l\varepsilon)$ and $\widetilde{Y}_{k_r}^\varepsilon = Y_{k_r}^\varepsilon \cap \Omega = \langle k_r\varepsilon, \omega_r)$.

For x in an inner cell Y_k^ε, the mapping τ^ε maps point (x, y) to $\varepsilon k + \varepsilon y$, while for x in the boundary uncomplete cells $\widetilde{Y}_{k_l-1}^\varepsilon$ and $\widetilde{Y}_{k_r}^\varepsilon$ the transform simply erase y coordinate, i.e. maps (x, y) to x.

Using function integer part $[x]$ of real number x defined by conditions $[x] \in \mathbb{Z}$, $x-1 < [x] \leq x$, the mapping τ^ε can be written as:

$$\tau^\varepsilon(x,y) = \begin{cases} \varepsilon \left[\dfrac{x}{\varepsilon}\right] + \varepsilon \cdot y & \text{for } \ x \in Y_k^\varepsilon, \ k = k_l, \ldots, k_r - 1 \ \ \text{(inner cells)} \\ x & \text{for } \ x \in \widetilde{Y}_k^\varepsilon \ k = k_l - 1, k_r \ \ \text{(boundary cells)} \end{cases} . \tag{2}$$

The two-scale-transform $T^\varepsilon : L^1(\Omega) \to L^1(\Omega \times Y)$ is then defined simply by

$$T^\varepsilon : u^\varepsilon(x) \mapsto \widehat{u^\varepsilon} \equiv (T^\varepsilon\, u^\varepsilon)(x,y) = u^\varepsilon(\tau^\varepsilon(x,y))\,. \tag{3}$$

4.2 Two-scale transform – multidimensional case

Ideas of 1D-case can be extended to multidimensional case. Let Y^ε be ε-scaled unit cell Y and for $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ and Y_k^ε be the scaled cell Y^ε shifted by vector $\varepsilon k = (\varepsilon k_1, \ldots, \varepsilon k_N)$. We distinguish complete inner and incomplete boundary ε-cells of the domain Ω. Then the two-scale mapping τ^ε is defined:

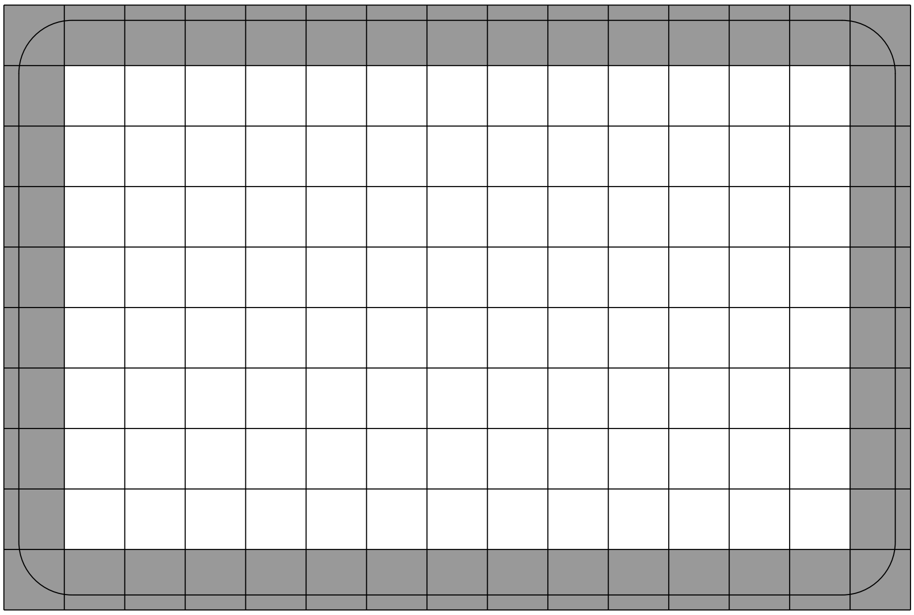

Figure 3: Inner and boundary cells

Definition 2 *Two-scale mapping $\tau^\varepsilon : \Omega \times Y \to \Omega$ in the notation introduced above is given by:*

$$\tau^\varepsilon(x,y) = \begin{cases} \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y & \textit{if } x \in Y_k^\varepsilon \subset \Omega \ \ \textit{(inner cells)} \\ x & \textit{if } x \in \widetilde{Y}^\varepsilon \ \ \textit{(boundary cells)} \end{cases} \tag{4}$$

and the two-scale-transform $T^\varepsilon : L^1(\Omega) \to L^1(\Omega \times Y)$ is defined by

$$T^\varepsilon : u^\varepsilon(x) \mapsto \widehat{u^\varepsilon} \equiv (T^\varepsilon\, u^\varepsilon)(x,y) = u^\varepsilon(\tau^\varepsilon(x,y))\,. \tag{5}$$

4.3 Measure preserving property

Since in both 1-D and multidimensional cases the unit cell Y has unit measure, the two-scale mapping τ^ε preserve measure: Let M be a measurable subset of Ω, then its full inverse image has the same measure as M. Thus also for any integrable function $u(x)$ we have $\iint_{\Omega \times Y} (T^\varepsilon u)(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega u(x)\, \mathrm{d}x$.

4.4 Alternative definition of the two-scale convergence

Definition 3 *Let $\{\varepsilon\}$ be a scale and $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$.*

We say that a sequence u^ε in $L^p(\Omega)$ (weakly) two-scale converges *(with respect to the scale $\{\varepsilon\}$ in $L^p(\Omega)$) to the limit $u_0(x,y) \in L^p(\Omega \times Y)$ iff*

$$\widehat{u^\varepsilon} = (T^\varepsilon u^\varepsilon) \quad \textit{converge to} \quad u_0 \quad \textit{in} \quad L^p(\Omega \times Y) \quad \textit{weakly.}$$

We say that a sequence u^ε in $L^p(\Omega)$ strongly two-scale converges *(with respect to the scale $\{\varepsilon\}$) to a limit $u_0(x,y) \in L^p(\Omega \times Y)$ iff*

$$\widehat{u^\varepsilon} = (T^\varepsilon u^\varepsilon) \quad \textit{converge to} \quad u_0 \quad \textit{in} \quad L^p(\Omega \times Y) \quad \textit{strongly.}$$

5 Basic properties

Lemma 4 *Let $\{\varepsilon\}$ be a scale and u^ε a sequence in $L^p(\Omega)$ and $u_0 \in L^p(\Omega \times Y)$. Then:*

(a) *If the sequence u^ε is two-scale converging (weakly or strongly), then it is bounded in $L^p(\Omega)$.*

(b) *If the two-scale limit u_0 exists, then it is unique (as an element of $L^p(\Omega \times Y)$).*

(c) *If the sequence u^ε two-scale converges (weakly or strongly) to u_0 with respect to the scale $\{\varepsilon\}$, then each its subsequence $\{\varepsilon'\}$ also two-scale converges (weakly or strongly) to the same limit u_0 but with respect to the subscale $\{\varepsilon'\}$.*

(d) *If the sequence $\{u^\varepsilon\}$ strongly two-scale converges to u_0, then it also (weakly) two-scale converges to the same limit u_0.*

(e) *If the sequence $\{u^\varepsilon\}$ two-scale converges to $u_0(x,y)$, then it also converge in $L^p(\Omega)$ weakly to $u^* \in L^p(\Omega)$ defined by $u^*(x) = \int_Y u_0(x,y)\,\mathrm{d}y$.*

(f) *If the sequence $\{u^\varepsilon\}$ strongly two-scale converges to u_0, then it also (weakly) two-scale converges to the same limit u_0 (with respect to the same scale).*

(g) *If the sequence u^ε (strongly) converges to u^* in $L^p(\Omega)$, then it also two-scale converges (both weakly and strongly) to $u_0(x,y) = u^*(x)$ with respect to the scale $\{\varepsilon\}$ and also with respect to each its subscale $\{\varepsilon'\}$.*

Remarks 5 *Let $f, g \in L^p(\Omega)$ and Y-periodic $\psi \in L^\infty(Y_\#)$ such that $\int_Y \psi(y)\,\mathrm{d}y = 0$. Then the sequence $\{u^\varepsilon\}$ defined by*

$$u^\varepsilon = f(x)\psi\left(\frac{x}{\varepsilon}\right) + g(x)$$

is bounded in $L^p(\Omega)$ and strongly two-scale converge to $f(x)\psi(y) + g(x)$ in $L^p(\Omega)$ with respect to the scale $\{\varepsilon\}$. Naturally it also converges weakly in $L^p(\Omega)$ to $g(x)$.

One can see that the local oscillations of u^ε are conserved in the two-scale limit, while it is lost in the ordinary weak $L^p(\Omega)$ limit. It was caused by the fact that the period of $\psi(x/\varepsilon)$ was "in" resonance with the scale $\{\varepsilon\}$.

If the period and the scale are not in resonance i.e. their ratio is neither rational nor constant, e.g. $\psi(x/\sqrt{2})$, then the sequence u^ε two-scale converge only weakly and the limit is independent of y, i.e. in the limit the local oscillations are lost.

Theorem 6 (Compactness) *Let $\{\varepsilon\}$ be a scale and $\{u^\varepsilon\}$ a bounded sequence in $L^p(\Omega)$. Then there exists a subscale $\{\varepsilon'\} \subset \{\varepsilon\}$ and a limit $u^* \in L^p(\Omega \times Y)$ such that $u^{\varepsilon'}$ (weakly) two-scale converge to u^* with respect to the subscale $\{\varepsilon'\}$ (in case $p = \infty$ weakly* two-scale converge).*

The proof is a direct consequence of compactness of bounded sequences in $L^p(\Omega \times Y)$.

Theorem 7 (Main convergence result) *Let a sequence u^ε two-scale converge to u_0 and sequence v^ε strongly two-scale converges to v^ε, both with respect to the same scale $\{\varepsilon\}$, the former in $L^p(\Omega)$ and the latter in $L^q(\Omega)$, where the exponents $p, q, r \geq 1$ are supposed to satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$.*

Then the product $u^\varepsilon v^\varepsilon$ two-scale converges to the limit $u_0 v_0 \equiv u_0(x,y) v_0(x,y)$ in $L^r(\Omega)$. Particularly for any $\varphi \in L^s(\Omega)$ with $s \in \langle 1, \infty \rangle$ satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$ we have

$$\int_\Omega u^\varepsilon(x)\, v^\varepsilon(x) \varphi(x)\, \mathrm{d}x \longrightarrow \iint_{\Omega \times Y} u_0(x,y)\, v_0(x,y) \varphi(x)\, \mathrm{d}x\, \mathrm{d}y\,.$$

6 Conclusion

The two-scale convergence in many cases enables to pass to the limits, see Theorem 7, particularly in the homogenization problems where the coefficients strongly two-scale converge. The alternative approach in Definition 3 removes the restriction on test functions and simplifies the proofs.

Moreover, using the idea of two scale mapping or two-scale transform only, the two-scale convergence can be generalized to non-periodic or even stochastic cases and extends thus homogenization to non-periodic and stochastic media.

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