An Approximation of Smooth Functions in one Variable

(Abstract)

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In this paper, we investigate the properties of a local high–order approximation of a smooth function $f(x)$ in a given interval $\langle a, b\rangle$. We assume that there exists a mesh $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step $h = \max_{1\leq i\leq n}(x_i - x_{i-1})$ and that the values $f_0, f_1, \ldots f_n$ of $f(x)$ in the given nodes $x_0, x_1, \ldots, x_n$ are known only. We derive classical second–order approximations $Df(x_i)$ of the derivatives $f'(x_i)$ and approximate the given function $f(x)$ by the Hermite interpolation cubic spline $\mathcal{H}_h[f](x)$ consisting of cubic polynomials determined by the values f_{i-1}, f_i and the derivatives $Df(x_{i-1}), Df(x_i)$ on the subintervals (x_{i-1}, x_i). We say that the mesh $a = x_0 < x_1 < \cdots < x_n = b$ is regular whenever $\max_{i,j}(x_i - x_{i-1})/(x_j - x_{j-1})$ is bounded by a constant independent of h. For an arbitrary function $f \in C^3\langle a, b\rangle$ we find positive constants C_1, C_2 such that

$$\|f - \mathcal{H}_h[f]\|_0 < C_1 h^3 |f|_{3,\infty} \text{ and } |f - \mathcal{H}_h[f]|_1 < C_2 h^2 |f|_{3,\infty}$$

is valid for all regular meshes. We illustrate numerically that the above orders of the error $f - \mathcal{H}_h[f]$ are optimal. We compare this result with the corresponding known results concerning intepolation linear and cubic splines. It turns out that the orders of the $L_2(a, b)$– and $H^1(a, b)$–norms of error of the interpolation by cubic splines are higher. On the other hand, the construction of the interpolant $\mathcal{H}_h[f]$ is local whereas the construction of the interpolation cubic spline requires to solve a global system of linear equations.

This investigation is a technically simple analogue of the corresponding high–order approximations of partial derivatives of smooth functions in two variables in the vertices of planar triangulations. This research is focused on the applications in post processing and in the control of local refinements of planar triangulations within the process of the finite element approximation of solutions of differential boundary–value problems.

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