An Approximation of Smooth Functions in one Variable

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Abstract

For the given values $f_0, f_1, \dots, f_n$ of a smooth function $f(x)$ in the nodes $a = x_0 < x_1 < \cdots < x_n = b$ and for the second–order approximations $Df(x_i)$ of the derivatives $f'(x_i)$, we approximate the function f on $\langle a, b\rangle$ by a function $\mathcal{H}_h[f] \in C^1\langle a, b\rangle$ which is a cubic polynomial determined by the data $f_{i-1}, Df(x_{i-1}), f_i, Df(x_i)$ on every subinterval $\langle x_{i-1}, x_i\rangle$. We find the orders of the error $f - \mathcal{H}_h[f]$ in the L_2 and H^1 norms, illustrate their optimality numerically and compare them with the corresponding norms of error of the interpolation of f by the linear and cubic splines.

1 Introduction

The aim of this paper is to analyze the properties of a local high–order approximation of a smooth function $f(x)$ in a given interval $\langle a, b\rangle$ under the assumption that the values $f_0, f_1, \dots f_n$ of $f(x)$ in the given nodes $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step h are known only. We derive well–known second–order approximations $Df(x_i)$ of the derivatives $f'(x_i)$ and approximate the given function $f(x)$ by the Hermite cubic spline $\mathcal{H}_h[f](x)$ consisting of cubic polynomials determined by the values f_{i-1}, f_i and the derivatives $Df(x_{i-1}), Df(x_i)$ on every subinterval (x_{i-1}, x_i). For so–called regular meshes we find positive constants C_1, C_2 such that

$$\|f - \mathcal{H}_h[f]\|_0 < C_1 h^3 |f|_{3,\infty} \quad \text{and} \quad |f - \mathcal{H}_h[f]|_1 < C_2 h^2 |f|_{3,\infty}$$

and compare these orders of error with the orders of error of intepolation by linear and cubic splines. Numerical experiments illustrate that the above orders of error are optimal.

This investigation is a technically simple analogue of the corresponding approximations of smooth functions in two variables in the vertices of triangulations with applications in post processing and in the control of local refinements of planar triangulations within the process of the finite element approximation of solutions of differential boundary–value problems. See [3], [1], [2] or [9].

The symbols $\|f\|_0$, $|f|_1$ and $|f|_{3,\infty}$ mean the $L_2(a, b)$–norm, $H^1(a, b)$–seminorm of f and the Chebyshev norm of f''' in $C\langle a, b\rangle$, consecutively.

2 A second–order approximation of the derivative

In this section, we present classical second–order approximations of the derivatives $f'(x_i)$ of smooth functions in the nodes and estimates of their errors in an informal way.

Let us consider the *nodes* $x_1 < x_2 < x_3$. For every function $f \in C\langle x_1, x_3\rangle$,

$$L(x) = f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) \tag{1}$$

with

$$\begin{aligned} L_1(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \\ L_2(x) &= \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ L_3(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \end{aligned}$$

is the *Lagrange interpolation polynomial* of f in the nodes x_1, x_2, x_3. In what follows, let us assume that $f \in C^3\langle x_1, x_3\rangle$. Then it is well–known that for every $x \in (x_1, x_3)$ there exists $\xi \in (x_1, x_3)$ such that

$$f(x) = L(x) + \frac{f'''(\xi)}{3!}(x-x_1)(x-x_2)(x-x_3). \tag{2}$$

The standard proof of (2) is based on the Rolle Theorem. We derive an analogical relation between the values $f'(x_2)$ and $L'(x_2)$ by the standard arguments from the proof of (2).

As $f(x) - L(x) = 0$ for $x = x_1, x_2, x_3$, there exist $\xi_1 \in (x_1, x_2)$ and $\xi_2 \in (x_2, x_3)$ such that

$$f'(x) - L'(x) = 0 \quad \text{for} \quad x = \xi_1, \xi_2$$

by the Rolle Theorem. Because the function

$$\varphi(t) = f'(t) - L'(t) - \frac{f'(x_2) - L'(x_2)}{(x_2-\xi_1)(x_2-\xi_2)}(t-\xi_1)(t-\xi_2)$$

satisfies $\varphi \in C^2\langle x_1, x_3\rangle$ and $\varphi(\xi_1) = \varphi(x_2) = \varphi(\xi_2) = 0$, there exist $\eta_1 \in (\xi_1, x_2)$, $\eta_2 \in (x_2, \xi_2)$ satisfying

$$\varphi'(\eta_1) = 0 = \varphi'(\eta_2)$$

and $\eta \in (\eta_1, \eta_2)$ such that $\varphi''(\eta) = 0$ by the Rolle Theorem again. As $L''' = 0$, we have

$$0 = \varphi''(\eta) = f'''(\eta) - 2\frac{f'(x_2) - L'(x_2)}{(x_2-\xi_1)(x_2-\xi_2)}$$

which is equivalent to

$$f'(x_2) = L'(x_2) + \frac{f'''(\eta)}{2}(x_2-\xi_1)(x_2-\xi_2). \tag{3}$$

Let us now assume that, for given nodes $a = x_0 < x_1 < \cdots < x_n = b$ with the *discretization step* $h = \max_{1\le i\le n}(x_i - x_{i-1})$, we know the values $f_i = f(x_i)$ for $i = 0, 1, \ldots, n$ of the function f only. If we approximate the unknown values $f(x)$ for $x \in (x_{i-1}, x_i)$ linearly, we obtain the linear–spline approximation $\mathcal{L}_h[f]$ of f, illustrated in Fig. 1.

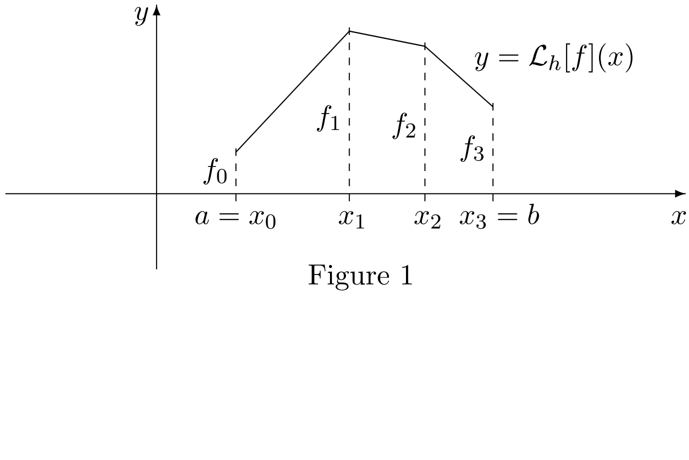

Figure 1

We can see that, in general, the function $\mathcal{L}_h[f]$ has no derivatives in the nodes x_i. If we want to use $\mathcal{L}_h[f]$ for an approximation of $f'(x_i)$ in an inner node x_i, of $f'(x_2)$ for example, we can use the value of the approximation of $f'(x)$ by $\mathcal{L}_h[f]'(x)$ for $x \in (x_1, x_2)$ (on the left from x_2) or by $\mathcal{L}_h[f]'(x)$ for $x \in (x_2, x_3)$ (on the right from x_2). We obtain the values $(f_2 - f_1)/(x_2 - x_1)$ or $(f_3 - f_2)/(x_3 - x_2)$ respectively. The accuracy of these approximations can be discovered by the following consideration using the Taylor Theorem.

As we have

$$\begin{aligned} f_1 &= f_2 + f'(x_2)(x_1 - x_2) + \frac{1}{2} f''(\zeta_1)(x_1 - x_2)^2, \\ f_3 &= f_2 + f'(x_2)(x_3 - x_2) + \frac{1}{2} f''(\zeta_2)(x_3 - x_2)^2, \end{aligned}$$

we obtain the approximations of $f'(x_2)$ in the forms

$$f'(x_2) = \frac{f_2 - f_1}{x_2 - x_1} + \frac{1}{2} f''(\zeta_1)(x_2 - x_1) \tag{4}$$

$$f'(x_2) = \frac{f_3 - f_2}{x_3 - x_2} - \frac{1}{2} f''(\zeta_2)(x_3 - x_2) \tag{5}$$

Due to $x_2 - x_1 < h$ and $x_3 - x_2 < h$, the sizes of errors of these approximations are proportional to the discretization step h. A fruitful idea is to approximate $f'(x_2)$ in form of a suitable linear combination of the approximations

$$\frac{f_2 - f_1}{x_2 - x_1} \quad \text{and} \quad \frac{f_3 - f_2}{x_3 - x_2}$$

from (4), (5) such that the error of the new approximation is of second–order (it is proportional to h^2). We can see from (3) that $L'(x_2)$ is a second–order approximation of $f'(x_2)$ because $|x_2 - \xi_1| < h$ and $|x_2 - \xi_2| < h$. A straightforward evaluation gives us

$$L'(x_2) = \alpha_1 \frac{f_2 - f_1}{x_2 - x_1} + \alpha_2 \frac{f_3 - f_2}{x_3 - x_2} \quad \text{for} \quad \alpha_1 = \frac{x_3 - x_2}{x_3 - x_1}, \quad \alpha_2 = \frac{x_2 - x_1}{x_3 - x_1},$$

so that

$$f'(x_2) \doteq Df(x_2) \equiv \alpha_1 \frac{f_2 - f_1}{x_2 - x_1} + \alpha_2 \frac{f_3 - f_2}{x_3 - x_2} \tag{6}$$

for

$$\alpha_1 = \frac{x_3 - x_2}{x_3 - x_1}, \quad \alpha_2 = \frac{x_2 - x_1}{x_3 - x_1}$$

is a second–order approximation of $f'(x_2)$. In this way we approximate $f'(x_i)$ in the inner nodes $x_1, \ldots, x_{n-1}$ of our mesh.

Mutual exchange of the indices 1 and 2 in (3) leads to the second–order approximation

$$f'(x_1) \doteq Df(x_1) \equiv \beta_1 \frac{f_2 - f_1}{x_2 - x_1} + \beta_2 \frac{f_3 - f_2}{x_3 - x_2} \tag{7}$$

for

$$\beta_1 = 1 + \frac{x_2 - x_1}{x_3 - x_1}, \quad \beta_2 = \frac{x_1 - x_2}{x_3 - x_1}$$

of f' useful in the least node $x_0 = a$ and an exchange of 2 and 3 in (3) leads to the second–order approximation

$$f'(x_3) \doteq Df(x_3) \equiv \gamma_1 \frac{f_2 - f_1}{x_2 - x_1} + \gamma_2 \frac{f_3 - f_2}{x_3 - x_2} \tag{8}$$

for

$$\gamma_1 = \frac{x_2 - x_3}{x_3 - x_1}, \quad \gamma_2 = 1 + \frac{x_3 - x_2}{x_3 - x_1}$$

applicable in the largest node $x_n = b$.

3 Approximation of a function

In this section, we use the operator D, described in (6), (7), (8) for an accurate approximation of a smooth function whose values are given in a finite number of nodes.

Definition. For every mesh $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step h and for the values $f_i = f(x_i)$, $i = 0, 1, \ldots, n$ of a function $f \in C\langle a, b\rangle$, we denote by $\mathcal{H}_h[f]$ a function from $C^1\langle a, b\rangle$ such that

(a) $\mathcal{H}_h[f](x_i) = f_i$,

(b) $\mathcal{H}_h[f]'(x_i) = Df(x_i)$ for $i = 0, 1, \ldots, n$,

(c) $\mathcal{H}_h[f]$ is a polynomial of degree less than or equal to three on the interval $\langle x_{i-1}, x_i\rangle$ for $i = 1, \ldots, n$.

Functions $\mathcal{H}_h[f]$ with the above properties (a) – (c) are called Hermite cubic splines. According to [6], the function $\mathcal{H}_h[f]$ is determined uniquely. In this case, we cannot use the classical results characterizing the interpolation error of the Hermite interpolation from [4] or [6] because $f'(x_i) \neq \mathcal{H}_h[f]'(x_i)$. It is also impossible to use the sophisticated tools developed for the analysis of the error of the finite element approximations presented in [8] or [5] because the so–called unisolvability of the finite elements is required. This requirement means that the behaviour of the interpolant, in our case of $\mathcal{H}_h[f]$, on every subinterval $\langle x_{i-1}, x_i\rangle$ is determined by the data from the subinterval $\langle x_{i-1}, x_i\rangle$ only; this is not the case for the interpolant $\mathcal{H}_h[f]$. In the following Lemma 1, we express the differences $f - \mathcal{H}_h[f]$ and $f' - \mathcal{H}_h[f]'$ on an arbitrary subinterval $\langle x_{i-1}, x_i\rangle$.

Lemma 1. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a mesh with the discretization step h and let $f \in C^3\langle a, b\rangle$. Let us choose a fixed index $i \in \{1, \ldots, n\}$, denote $I = \langle x_{i-1}, x_i\rangle$ and put $k = x_i - x_{i-1}$, $f'_{i-1} = f'(x_{i-1})$, $f'_i = f'(x_i)$. Then there exist parameters c_0, c_1, c_2 satisfying $|c_0| < |f|_{3,\infty}/2$, $|c_1| < |f|_{3,\infty}/2$ and $|c_2| < |f|_{3,\infty}/4$ such that the following statements (a), (b) are valid.

(a) For every $x \in I$ there exist $\eta_1, \eta_2 \in I$ and η between η_1, η_2 satisfying

$$\begin{aligned}(\mathcal{H}_h[f] - f)(x) &= (x - x_{i-1})c_0hk + (x - x_{i-1})^2\left[(\eta_2 - \eta_1)f'''(\eta)/2 - c_0h\right]\\ &+ (x - x_{i-1})^2(x - x_i)\left[c_2 + h(c_0 + c_1)/k\right].\end{aligned}$$

(b) For every $x \in I$ there exist $\eta_1, \eta_2 \in I$ and η between η_1, η_2 satisfying

$$\begin{aligned}(\mathcal{H}_h[f] - f)'(x) &= (x - x_{i-1})\left[f'''(\eta)(\eta_2 - \eta_1) - 2c_0h\right]\\ &+ (x - x_{i-1})(3x - 2x_i - x_{i-1})\left[c_2 + h(c_0 + c_1)/k\right].\end{aligned}$$

Proof. (i) There exist c_0, c_1 such that $|c_0| < |f|_{3,\infty}/2$, $|c_1| < |f|_{3,\infty}/2$ and

$$f'(x_{i-1}) - Df(x_{i-1}) = c_0hk, \quad f'(x_i) - Df(x_i) = c_1hk\ :$$

Due to (3), we have $|f'(x_{i-1}) - Df(x_{i-1})| < |f|_{3,\infty}hk/2$. The first statement follows immediately and the proof of the second statement is analogical.

(ii) There exists c_2 such that $|c_2| < |f|_{3,\infty}/4$ and

$$f'_{i-1} + f'_i - 2\frac{f_i - f_{i-1}}{k} = c_2k^2\ :$$

Due to the Taylor Theorem, there exist $x_{i-1} < \xi_1 < \xi_2 < x_i$ such that

$$\begin{aligned} f(x_i - k/2) &= f_{i-1} + f'_{i-1}k/2 + f''(\xi_1)k^2/8 \\ f(x_i - k/2) &= f_i - f'_i k/2 + f''(\xi_2)k^2/8 \end{aligned}$$

If we subtract the second identity from the first one, multiply the difference by $2/k$ and use the relation $f''(\xi_1) - f''(\xi_2) = (\xi_1 - \xi_2)f'''(\xi)$ for a suitable $\xi \in (\xi_1, \xi_2)$ due to the Mean Value Theorem, we obtain the statement.

Now, we express the difference $\mathcal{H}_h[f] - f$. The definition of $\mathcal{H}_h[f]$ and (i) tell us that the restriction of the cubic polynomial $\mathcal{H}_h[f]$ on the interval I is determined by the conditions

$$\begin{array}{ll} \mathcal{H}_h[f](x_{i-1}) = f_{i-1} & \mathcal{H}_h[f]'(x_{i-1}) = f'_{i-1} + c_0 hk \\ \mathcal{H}_h[f](x_i) = f_i & \mathcal{H}_h[f]'(x_i) = f'_i + c_1 hk \end{array}$$

The well–known construction of the polynomial $\mathcal{H}_h[f]$, see [4] or [6], leads to the formula

$$\begin{aligned} \mathcal{H}_h[f](x) &= f_{i-1} + (x - x_{i-1})(f'_{i-1} + c_0 hk) \\ &+ (x - x_{i-1})^2 \frac{f_i - f_{i-1} - k(f'_{i-1} + c_0 hk)}{k^2} \\ &+ (x - x_{i-1})^2 (x - x_i) \frac{f'_{i-1} + f'_i - 2(f_i - f_{i-1})/k + (c_0 + c_1)hk}{k^2}. \end{aligned} \tag{9}$$

If we use the identities

$$f_{i-1} + (x - x_{i-1})f'_{i-1} = f(x) - (x - x_{i-1})^2 f''(\eta_1)/2$$

for some $\eta_1 \in I$ due to the Taylor Theorem,

$$f_i - f_{i-1} - kf'_{i-1} = f''(\eta_2)k^2/2 \tag{10}$$

for some $\eta_2 \in I$ due to (5) and the statement (ii) then we obtain

$$\begin{aligned} \mathcal{H}_h[f](x) &= f(x) + (x - x_{i-1})c_0 hk \\ &+ (x - x_{i-1})^2 \left[(f''(\eta_2) - f''(\eta_1))/2 - c_0 h \right] \\ &+ (x - x_{i-1})^2 (x - x_i) \left[c_2 + h(c_0 + c_1)/k \right]. \end{aligned}$$

This and the Mean Value Theorem give us

$$\begin{aligned} (\mathcal{H}_h[f] - f)(x) &= (x - x_{i-1})c_0 hk + (x - x_{i-1})^2 \left[(\eta_2 - \eta_1)f'''(\eta)/2 - c_0 h \right] \\ &+ (x - x_{i-1})^2 (x - x_i) \left[c_2 + h(c_0 + c_1)/k \right] \end{aligned}$$

for suitable $\eta_1, \eta_2 \in I$ and η between η_1, η_2.

In order to derive the difference $\mathcal{H}_h[f]' - f'$, we find the following derivative of (9).

$$\begin{aligned} \mathcal{H}_h[f]'(x) &= f'_{i-1} + c_0 hk + 2(x - x_{i-1}) \frac{f_i - f_{i-1} - k(f'_{i-1} + c_0 hk)}{k^2} \\ &+ (x - x_{i-1})(3x - 2x_i - x_{i-1}) \frac{f'_{i-1} + f'_i - 2(f_i - f_{i-1})/k + (c_0 + c_1)hk}{k^2} \end{aligned}$$

If we use (10), (ii) and the identity

$$f'_{i-1} = f'(x) - f''(\eta_1)(x - x_{i-1})$$

valid for a suitable $\eta_1 \in (x_{i-1}, x)$ by the Mean Value Theorem, we obtain

$$\begin{aligned}(\mathcal{H}_h[f]-f)'(x) &= (x-x_{i-1})\left[f'''(\eta)(\eta_2-\eta_1)-2c_0h\right] \\ &+ (x-x_{i-1})(3x-2x_i-x_{i-1})\left[c_2+h(c_0+c_1)/k\right]. \quad \square\end{aligned}$$

In Theorem 1, we prove the accurracy of the approximation of functions $f \in C^3\langle a,b\rangle$ by the functions $\mathcal{H}_h[f]$ in the L_2–norm and the H^1–seminorm under the assumption that the mesh of the interval $\langle a,b\rangle$ is regular.

Definition. Let $\nu \in (0,1)$. We say that the mesh $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step h is *regular* whenever $x_i - x_{i-1} \ge \nu h$ for $i = 1, \ldots, n$.

Theorem 1. For arbitrary regular mesh $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step h and for every function $f \in C^3\langle a,b\rangle$, we have

$$\|\mathcal{H}_h[f]-f\|_0 \le C_1 h^3 |f|_{3,\infty} \quad \text{for} \quad C_1 = \sqrt{\frac{583(b-a)}{840\nu}}$$

and

$$|f-\mathcal{H}_h[f]|_1 \le C_2 h^2 |f|_{3,\infty} \quad \text{for} \quad C_2 = \sqrt{\frac{47(b-a)}{24\nu}}.$$

Proof. The difference $\mathcal{H}_h[f]-f$ from Lemma 1 gives us

$$\|\mathcal{H}_h[f]-f\|_{0,I}^2 = \int_I (\mathcal{H}_h[f]-f)^2(x)\,dx = I_1 + \cdots + I_6.$$

It is easy to see that

$$\begin{aligned}
|I_1| &= \int_I (x-x_{i-1})^2 c_0^2 h^2 k^2 dx \le h^2 k^5 \frac{|f|_{3,\infty}^2}{12} \\
|I_2| &= \left|\int_I (x-x_{i-1})^3 c_0 hk \left[(\eta_2-\eta_1)f'''(\eta)-2c_0h\right] dx\right| \le (h+k)hk^5 \frac{|f|_{3,\infty}^2}{8} \\
|I_3| &= \left|\int_I (x-x_{i-1})^4 \left[(\eta_2-\eta_1)\frac{f'''(\eta)}{2}-c_0h\right] dx\right| \le (h+k)^2 k^5 \frac{|f|_{3,\infty}^2}{20} \\
|I_4| &= 2\left|\int_I (x-x_{i-1})^3 (x-x_i) c_0 hk \left[c_2+\frac{h}{k}(c_0+c_1)\right] dx\right| \\
&\le \left(h+\frac{k}{4}\right) hk^5 \frac{|f|_{3,\infty}^2}{20} \\
|I_5| &= \left|\int_I (x-x_{i-1})^4 (x-x_i)\left[(\eta_2-\eta_1)f'''(\eta)-2c_0h\right]\left[c_2+\frac{h}{k}(c_0+c_1)\right] dx\right| \\
&\le (h+k)\left(h+\frac{k}{4}\right) k^5 \frac{|f|_{3,\infty}^2}{30} \\
|I_6| &= \int_I (x-x_{i-1})^4 (x-x_i)^2 \left[c_2+\frac{h}{k}(c_0+c_1)\right]^2 dx \le \left(h+\frac{k}{4}\right)^2 k^5 \frac{|f|_{3,\infty}^2}{105}
\end{aligned}$$

We increase the above estimates of $\|\mathcal{H}_h[f]-f\|_{0,I}^2$ by putting $k = h$. If we compute the sum of these terms and use the fact that $n \le (b-a)/(\nu h)$, we obtain

$$\|\mathcal{H}_h[f]-f\|_0^2 \le n\frac{583}{840} h^7 |f|_{3,\infty}^2 \le \frac{583(b-a)}{840\nu} h^6 |f|_{3,\infty}^2.$$

By means of the same technique, we obtain

$$|\mathcal{H}_h[f]-f|_1^2 \leq \frac{47(b-a)}{24\nu} h^4 |f|_{3,\infty}^2. \quad \square$$

In the following example, we illustrate that the orders of the interpolation error measured in the norms from Theorem 1 are optimal.

Example. Let us consider the function

$$f(x) = \frac{1}{1+x^2}$$

on the interval $\langle -1, 2\rangle$. In the following Table 1, we present the values of the norms $\|\mathcal{H}_h[f] - f\|_0$ and $|f - \mathcal{H}_h[f]|_1$ for equidistant meshes on $\langle -1, 2\rangle$ with the discretization steps $h = 3/2^2, 3/2^3, \ldots, 3/2^7$.

h	$\|\mathcal{H}_h[f]-f\|_0$	$\vert\mathcal{H}_h[f]-f\vert_1$
0.75	0.63308 e-1	0.14304
0.375	0.52994 e-2	0.32594 e-1
0.1875	0.43326 e-3	0.79324 e-2
0.09375	0.43908 e-4	0.19512 e-2
0.046875	0.51037 e-5	0.48421 e-3
0.0234375	0.63335 e-6	0.12086 e-3

Table 1

For comparison, we summarize the known results concerning the orders of the errors of interpolation of smooth functions by linear and cubic splines.

Definition. For arbitrary mesh $a = x_0 < x_1 < \cdots < x_n = b$ with the discretization step h and for a function $f \in C\langle a, b\rangle$, we denote by $\mathcal{S}_h[f]$ the interpolation cubic spline of f in the nodes $x_0, \ldots, x_n$ such that

$$f'(a) = \mathcal{S}_h[f]'(a) \text{ and } f'(b) = \mathcal{S}_h[f]'(b) \quad \text{or} \quad f''(a) = \mathcal{S}_h[f]''(a) \text{ and } f''(b) = \mathcal{S}_h[f]''(b).$$

Theorem 2. For every function $f \in C^2\langle a, b\rangle$ $[f \in C^4\langle a, b\rangle]$ there exist constants C_1, C_2 such that

$$\begin{array}{ccc} \|\mathcal{L}_h[f]-f\|_0 < C_1 h^2 & \text{and} & |\mathcal{L}_h[f]-f|_1 < C_2 h \\ \big[\|\mathcal{S}_h[f]-f\|_0 < C_1 h^4 & \text{and} & |\mathcal{S}_h[f]-f|_1 < C_2 h^3\big] \end{array}$$

for every regular mesh with the discretization step h.

Proof. The statement concerning the linear splines $\mathcal{L}_h[f]$ can be found in [11] and the statement concerning the cubic splines has been proved in [10].

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