Actions of Join Spaces of Continuous Functions on Hypergroups of Second-Order Linear Differential Operators

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Dedicated to Professor František Neuman on the occasion of his 70th birthday

Introduction

One from classical constructions of algebraic binary hyperstructures (semihypergroups and hypergroups) from ordered algebraic systems is based on a certain Lemma on principal ends generated by products of pairs of elements – shortly termed as Ends-Lemma. First version of this auxiliary lemma was obtained in the monography [12]. The mentioned lemma is applied not only in case of hyperstructures created from ordered or quasi-ordered semigroups and groups but also in constructions based centralizers of various transformations (in particular endomorphism monoids of mono-unary algebras of various functions or operators). Moreover constructions of this type yield the way to obtain semihypergroups and hypergroups of linear differential operators – ordinary differential operators of a given order n or partial differential operators and mentioned constructions allow also to create hyperstructures of integral operators of Fredholm or Volterra-type – cf. [23].

In the present contribution we construct actions of commutative transposition hypergroups i.e. join spaces created from rings of continuous and smooth functions of a given class on semihypergroups or hypergroups of second order linear ordinary differential operators. These constructed structures are in fact discrete dynamical systems with a phase (additive) hypergroups of continuous and smooth functions and phase set formed by the above mentioned differential operators. As a suitable synonyma we can use a multiautomaton without output function, where the transition function or next state function satisfies so called Generalized Mixed Asociativity Condition (GMAC) – see bellow. It is to be noted that the systematic investigation of transformations of ordinary differential equations using algebraic tools and methods is going back into fifties which has been initiated by Professor Otakar Borůvka and his school. In this connection recall from the paper [34] of one of his outstanding successors – Professor František Neuman, to whom we dedicate our contribution: "Algebraic, topological and geometrical tools together with the methods of the theory of dynamical systems and functional equations make it possible to deal with problems concerning global properties of solutions by contrast to the previous local investigations and isolated results". The main part (4) of this contribution contains the isomorphism theorem between actions (discrete dynamical systems – called in the last Borůvka's paper [5] as algebraic spaces with operators) of an additive join space of real functions of the given class C^k on the phase space formed by second-order linear ordinary differential operators and actions of the mentioned functional join space of hypergroups of transformation operators which are restrictions on intervals of the real line of Laplace-transform images of the Volterra-type integral operators with difference kernels. Notice that the just mentioned classical Laplace transform, which is a part of curricula of mathematical subjects – in particular at faculties of electrical engineering of universities of technology, enables the construction of an embedding of the above mentioned Volterra integral operators in the hypergroup of certain complex transformations generalizing affine transformations of vector function spaces. These facts serve as motivating factors of presented constructions.

1. Preliminaries. Recall some basic notions a notation of the hypergroup theory — for further information cf.[6], [7], [9–19], [27–33]. A *hypergroupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet\colon H \times H \to \mathcal{P}^*(H)$ is binary hyperoperation on H. Symbol $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a *semihypergroup.* If moreover the reproduction axiom $a \bullet H = H = H \bullet a$ is satisfied for any element $a \in H$, then the pair $(H, \bullet)$ is called a *hypergroup.*

We can define a *hyperproduct* of any pair of nonempty subsets of $A, B \in H$ as $A \bullet B = \cup\{a \bullet ba \in A, b \in B\}$. A *subhypergroupoid* of a hypergroupoid $(H, \bullet)$ is a pair $(S, \bullet)$, where $S \bullet S \subseteq S$, i.e. the set S is multiplicatively closed. If the subhypergroupoid $(S, \bullet)$ of $(H, \bullet)$ is a hypergroup, then it is called a *subhypergroup* of $(H, \bullet)$. A hypergroup $(H, \bullet)$ is called a transposition hypergroup or a join space if it satisfies the transopisition axiom:
For $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a \bullet d \approx b \bullet c$ (here $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$), where sets $b\backslash a = \{x \in H; a \in b \bullet x\}$, $c/d = \{x \in H; c \in x \bullet d\}$ are called left and right extensions or functions in the given order. Clearly, if the hyperooperation "$\bullet$" is commutative then fractions $b\backslash a$, a/b coincide.

We describe a simple but important construction from [12], which has been used also in [14], [17], [19], [20], [25] or [26] and which enables us to obtain a certain sense analogous results to those presented in contribution [16]. In what follows $\mathbb{N} = \{1, 2 \dots\}$ is a set of all positive integers (natural numbers) and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By a *quasi-ordered semigroup* we mean a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a semigroup and binary relation $\leq$ is a quasi-ordering (i.e is reflexive and transitive) on the set G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq y \bullet z$ and $z \bullet x \leq z \bullet y$ hold. By an *ordered* (*semi*) *group* we mean (as usual) a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a (semi)group and $\leq$ is a reflexive, antisymmetrical and transitive binary relation on G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq z$ and $z \bullet x \leq z \bullet y$ hold. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is a principal and generated by $a \in G$. To any element a of a noncommutative group there is assigned a pair of mappings $\lambda_a : G \to G$, $\rho_a\colon G \to G$ defined by $\lambda_a(x) = a \bullet x$, $\rho_a(x) = x \bullet a$. These are called *left* and *right translation determined by* $a \in G$ respectively. Notice that a group with an ordering $(G, \bullet, \leq)$ is an ordered group if and only if all its left and right translations λ_a, ρ_a, $a \in S$ are order-preserving, i.e. isotone selfmaps of the ordered set $(S, \leq)$. By an *inclusion homomorphism* we mean a mapping $f\colon (G, \bullet_G) \to (H, \bullet_H)$ such that $f(a \bullet_H b) \subset f(a) \bullet_G f(b)$ for all pairs $a, b \in G$. If equalities hold instead of inclusion and the mapping f is bijective then f is an isomorphism and we write $(G, \bullet_G) \simeq (H, \bullet_H)$.

The following lemma, which is crucial for our considerations, is proved in [16] for the first time in [12], pp 146–7.

1.1 Lemma. *Let a triple $(G, \cdot, \leq)$ be a quasi-ordered semigroup. Define a hyperoperation*

$$*\colon G \times G \to \mathcal{P}^*(G) \quad \textit{by} \quad a * b = [a, b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in G$.

1. *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*

2. *Let $(G,*)$ be the above defined semihypergroup. Then $(G,*)$ is a hypergroup iff for any par of elements $a,b\in G$ there exists a pair of elements $c,c'\in G$ with a property $a\cdot c\le b$, $c'\cdot a\le b$.*

1.2 Remark. Notice that if $(G,\cdot,\le)$ is a quasi-ordered group then the condition stated under 2. is satisfied, hence the final hyperstructure is a hypergroup.

As a result of discussing the Laplace transform we consider the half-plane of complex numbers $\Omega=\{z,\mathrm{Re} z>0\}$ in [23]. For the purpose of this contribution let us consider $\mathbb{C}$ the field of all complex numbers, $\emptyset\ne\Omega\subseteq\mathbb{C}$ a compact (i.e. closed and bounded) subset, $\mathbb{C}^\Omega=\{f\colon\Omega\to\mathbb{C}\}$ a normed algebra of all complex functions of one variable. We define an operator $T(\lambda,F,\varphi)\colon\mathbb{C}^\Omega\to\mathbb{C}^\Omega$, where $\lambda\in\mathbb{C}$ and $F,\varphi\in\mathbb{C}^\Omega$ by

$$T(\lambda,F,\varphi)\big(f(z)\big)=\lambda F(z)\,f(z)+\varphi(z)$$

or, of the sake bravity

$$T(\lambda,F,\varphi)\,(f)=\lambda Ff+\varphi$$

for any $f\in\mathbb{C}^\Omega$ and any $z\in\mathbb{C}$. We denote $\mathcal{T}(\Omega)$ the set of all operators $T(\lambda,F,\varphi)$, i.e.

$$\mathcal{T}(\Omega)=\big\{T(\lambda,F,\varphi);\lambda\in\mathbb{C},F,\varphi\in\mathbb{C}^\Omega\big\}$$

For purpose of our consideration we restrict ourselves to operators $T(1,F,\varphi)$ restricted on real functions defined on intervals I of the real line. As usually, $\mathbb{C}^k(I)$ stands for the commutative ring of all real functions defined on an open interval I of reals, and having there continuous derivatives up to order $k\ge 0$; $\mathbb{C}_+(I)$ is the subsemiring of $\mathbb{C}(I)=\mathbb{C}^0(I)$ of all positive continuous functions defined on the interval I. Thus $T(1,p_1,q_1)\circ T(1,p_2,q_2)=T(1,p_1p_2,p_1q_2+q_1)$ and $T(1,p,q)\,(f)=pf+q$ for any $f\in\mathbb{C}^k(I)$. Evidently, if all functions $p:I\to\mathbb{R}$ determining $T(1,p,q)$ are positive, we obtain that $\mathbb{S}(I)$ is a (noncommutative) group.

Let a triad (S,S,δ), where $S\ne\emptyset$ is a set (which may be endowed with a special structure) be a multiautomaton with the phase (semi-)hypergroup H and transition function $\delta\colon S\times H\to S$ satisfying the *generalized mixed associativity condition* (abbr. as GMAC)

$$\delta\big(\delta(s,a),b\big)\in\delta(s,a\cdot b)\tag{1}$$

for any $s\in S$ and any pair of elements a, $b\in H$. Following a number of books and articles concerning actions of semigroups and groups we also say that the (*semi-*)*hypergroup H acts on the phase set* (also known as *phase space*) S or that the triad (S,H,δ) which satisfies the above mentioned general mixed associativity condition, is the *multi-act* (also known as *multiaction*) of H on S.

2. Hypergroups of operators.

Denote by $\mathbb{A}_2(I)$ the set of nonsingular ordinary linear homogeneous differential equations of the second order

$$y''+p(x)y'+q(x)y=0\,,\tag{2}$$

such that $p\in\mathbb{C}_+(I)$, $q\in\mathbb{C}(I)$. Let us denote by Id the identity operator $D=\frac{d}{dx}$. Further, we denote by $[p,q]$ as usually an ordered pair of functions p, q. By $L(p,q)$ will be denoted the differential operator $L(p,q)=D^2+p(x)D+q(x)\,\mathrm{Id}$, with the use of that the equation $\big(P_2(y,x;I)\big)$ has the form $L(p,q)(y)=0$. We denoted by

$$\mathbb{LA}_2(I)=\big\{L(p,q):\mathbb{C}^2(I)\to\mathbb{C}(I);[p,q]\in\mathbb{C}_+(I)\times\mathbb{C}(I)\big\}\tag{3}$$

the set of all such differential operators.

For $r\in\mathbb{R}$ we denote by $\chi_r\colon I\to\mathbb{R}$ the constant function with the value r.

All the following results were obtained in [14].

2.1 Proposition. *Let $I \subset \mathbb{R}$ be an open interval, $\mathbb{LA}_2(I) = \{L(p,q); p,q \in \mathbb{C}(I), p(x) > 0, x \in I\}$. For any pair of differential operators $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ define*

$$L(p_1,q_1) \cdot L(p_2,q_2) = L(p_1p_2, p_1q_2 + q_1)$$

and $L(p_1,q_1) \le L(p_2,q_2)$ if $p_1(x), q_1(x) \le q_2(x)$ for any $x \in I$. Then $(\mathbb{LA}_2(I), \cdot, \le)$ is a non-commutative ordered group with the unit element $L(\chi_1, \chi_0)$. □

Now we apply the simple construction of a hypergroup from Lemma 1 into this considered concrete case of differential operators:

For arbitrary pair of operators $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ we put

$$\begin{aligned} L(p_1q_1) * L(p_2,q_2) &= \{L(p,q) \in \mathbb{LA}_2(I); L(p_1,q_1) \cdot L(p_2,q_2) \le L(p,q)\} \\ &= \{L(p,q) \in \mathbb{LA}_2(I); L(p_1p_2, p_1q_2 + q_1) \le L(p,q)\} \\ &= \{L(p_1p_2,q); \in \mathbb{C}(I), p_1q_2 + q_1 \le q\}\,. \end{aligned}$$

Then we obtain from Proposition 2.1 with respect to Lemma 1.1 immediately:

2.2 Proposition. *Let $I \in \mathbb{R}$ be an open interval and let $* : \mathbb{LA}_2(I) \times \mathbb{LA}_2(I) \to \mathcal{P}^*(\mathbb{LA}_2))$ be the above defined binary hyperoperation. Then the hypergroupoid $(\mathbb{LA}_2(I), *)$ is an noncommutative hypergroup.* □

2.3 Theorem. *Let $I \subset \mathbb{R}$ be an open interval, $\mathbb{LA}_2(I) = \{L(p,q); [p,q] \in \mathbb{C}_+(I) \times \mathbb{C}(I)\}$ be the set of ordinary linear differential operators of second order — i.e. $L(p,q)(y) = y'' + p(x)y' + q(x)y = 0$, $y \in \mathbb{C}^2(I)$. If $L(p_1,q_1)*L(p_2,q_2) = \{L(p,q) \in \mathbb{LA}_2(I); p_1p_2 = p, p_1q_2+q_1 \le q\}$ for any pair $L(p_1q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ then $(\mathbb{LA}_2(I), *)$ is noncommutative transposition hypergroup, i.e. a noncommutative join space.* □

Similarly as above we define a binary hyperoperation $\bullet \colon \mathbb{S}(I) \times \mathbb{S}(I) \to \mathcal{P}(\mathbb{S}(I))$ by

$$\begin{aligned} T(1,p_1,q_1) \bullet T(1,p_2,q_2) = \{&T(1,u,v); u(x) = p_1(x)p_2(x), \\ &p_1(x)q_2(x) + q_1(x) \le v(x), x \in I\}\,. \end{aligned}$$

Defining a binary relation $\le$ on the group $\mathbb{S}(I)$ (functions p are positive) by $T(1,p_1,q_1) \le T(1,p_2,q_2)$ whenever $p_1(x) = p_2(x)$, $q_1(x) \le q_2(x)$ for all $x \in I$, we obtain that $(\mathbb{S}(I), \circ, \le)$ is an ordered group. Then by Lemma 1.1 the hypergroupoid $(\mathbb{S}(I), \bullet)$ is a (noncommutative) hypergroup. In fact, from the following theorem and Theorem 2.3 there follows that this hypergroup $(\mathbb{S}(I), \bullet)$ is transposition hypergroup, i.e. a noncommutative join space.

2.4 Theorem. *Let $I \subset \mathbb{R}$ be an open interval, $(\mathbb{LA}_2(I), *) = (\{L(p,q); p \in \mathbb{C}_+^k(I), q \in \mathbb{C}^k(I)\}, *\}$, $(\mathbb{S}(I), \bullet) = (\{T(1,p,q); p \in \mathbb{C}_+^k(I), q \in \mathbb{C}^k(I)\}, \bullet)$ be the above constructed hypergroups. Let*

$$\Phi \colon \mathbb{LA}_2(I) \to \mathbb{S}(I)$$

be a mapping defined by $\Phi(L(p,q)) = T(1,p,q)$ for any differential operator $L(p,q) \in \mathbb{LA}_2(I)$. Then $\Phi \colon (\mathbb{LA}_2(I), *) \to (\mathbb{S}(I), \bullet)$ is a hypergroup-isomorphism, consequently $(\mathbb{LA}_2(I), *) \cong (\mathbb{S}(I), \bullet)$. □

Proof. Suppose $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ are operators such that $F(L(p_1,q_1)) = F(L(p_2,q_2))$. Then $T(1,p_1,q_1)(f) = p_1f + q_1 = p_2f + q_2 = T(1,p_2,q_2)(f)$ for all functions $f \in \mathbb{C}^k(I)$. Thus considering $f(x) \equiv 1$, i.e. $f = \chi_1$ we get $q_1 = q_2$ and for $f(x) \equiv x$ we have $p_1 = p_2$, hence $L(p_1,q_1) = L(p_2,q_2)$. The mapping F is evidently also surjective, consequently it is a bijection. Now,

$$\begin{aligned} \Phi(L(p_1,q_1) \cdot L(p_2,q_2)) &= \Phi(L(p_1p_2, p_1q_2 + q_1)) \\ &= T(1, p_1p_2, p_1q_2 + q_1) = T(1,p_1,p_2) \circ T(1,p_2,q_2) \\ &= \Phi(L(p_1,p_2)) \circ F(L(p_2,q_2))\,, \end{aligned}$$

thus the mapping Φ is a group-homomorphism, as well. These facts imply that Φ is an isomorphism of the hypergroup $(\mathbb{LA}_2(I), *)$ onto the hypergroup $(\mathbb{S}(I), \bullet)$. $\square$

2.5 Example. Consider a collection $\mathcal{V}_2$ of two-dimensional vector spaces of real functions of the form

$$V(\varphi_V, \psi_V) = \{c_1\varphi_V + c_2\varphi_V; c_1, c_2 \in \mathbb{R}\},$$

where either

$$\varphi_V(x) = \exp(\lambda_V x), \quad \varphi_V(x) = \exp(\mu_V x), \tag{4}$$

$x \in \mathbb{R}$ and $\lambda_V + \mu_V < 0$, $\lambda_V \neq \mu_V$, or

$$\varphi_V(x) = \exp(\alpha_V c) \cos \beta_V x, \quad \varphi_V(x) = \exp(\alpha_V x) \sin \beta_V x, \tag{5}$$

with $\alpha_V < 0$, $x \in \mathbb{R}$, or

$$\varphi_V(x) = \exp(\lambda_V x), \quad \varphi(x) = x \exp(\lambda_V x) \tag{6}$$

with $\lambda_V < 0$, $x \in \mathbb{R}$.

The collection $\mathcal{V}_2$ of vector spaces $V(\varphi_V, \psi_V)$ (here pair of functions $\{\varphi_V, \psi_V\}$ forms a base of $V(\varphi_V, \psi_V)$) can be endowed with a binary operation $\varkappa\colon \mathcal{V}_2 \times \mathcal{V}_2 \to \mathcal{V}_2$ such that the pair $(\mathcal{V}_2, \varkappa)$ is a (non-commutative) group isomorphic to the group $\mathbb{R}^+ \times \mathbb{R}, \cdot$ where the binary operation "·" is defined by the rule

$$[a, b] \cdot [c, d] = [ac, ad + b], [a, b], [c, d] \in \mathbb{R}^+ \times \mathbb{R}.$$

The unit of this group is the pair $[1, 0]$ and inverse elements are of the form $[a, b]^{-1} = [a^{-1}, -a^{-1}b]$, $a, b \in \mathbb{R}$, $a > 0$. Indeed, the vector space with the base of the form (4) is the solution space of the second order homogeneous differential equation with constant coefficients of the form

$$y''(x) - (\lambda_V + \mu_V)y'(x) + \lambda_V \mu_V y(x) = 0$$

with $\lambda_V + \mu_V < 0$. If $U(\varphi_U, \psi_U)$ is another such vector space with the base

$$\varphi_U(x) = \exp(\lambda_U x), \quad \psi_U(x) = \exp(\mu_U x),$$

$\lambda_U + \mu_U < 0$, then the vector space

$$\varkappa\big(V(\varphi_V, \psi_V), U(\varphi_U, \psi_U)\big) = \{c_1\varphi + c_2\psi; c_1 c_2 \in \mathbb{R}\},$$

where $\{\varphi, \psi\}$ — a base of this space — is a fundamental system of solutions of this homogeneous equation

$$y''(x) + (\lambda_V + \mu_V)(\lambda_U + \mu_U)\, y'(x) + \big[(\lambda_V + \mu_V)\lambda_U \mu_U + \lambda_V \mu_V\big]\, y(x) = 0.$$

Description of the vector space $\varkappa\big(V(\varphi_V, \psi_V), U(\varphi_U, \psi_U)\big) = W(\varphi, \psi)$ in cases when spaces $V(\varphi_V, \psi_V)$, $U(\varphi_U, \psi_U)$ are of the type (5) or (6) or of mixed types let us left to the reader. Notice, that the group $(\mathcal{V}_\in, \cdot)$ is isomorphic to a subgroup of the group $(\mathbb{LA}_2(\mathbb{R}), \cdot)$ — exactly to the subgroup of all differential operators $L(a, b)$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. These groups are moreover isomorphic to the group of transformation operators $T(1, a, b)$, $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ with binary operation which is usual composition of mappings.

3. Additive hypergroups of continuous and smooth functions. Using the above mentioned simple constructions of the commutative hypergroup $(\mathbb{Z}, +)$ of all integers we obtain a sequence of commutative join hypergroups of smooth functions carriers of which are rings of functions of the class $\mathbb{C}^k$ for $k = 0, 1, 2, \ldots, \infty$.

Let $I \subset \mathbb{R}$ be an open interval and $\mathbb{C}^k(I)$ be the ring of functions $f: I \to \mathbb{R}$ of the class $\mathbb{C}^k$. For a pair of functions $f, g \in \mathbb{C}^k(I)$ — as usually — $f \leq g$ means that $f(x) \leq g(x)$ for each $x \in I$. So, for any pair of functions $f, g \in \mathbb{C}^k(I)$ define $f + g = [f+g)_{\leq} = \{h: I \to \mathbb{R}; f + g \leq h\}$. Then $+: \mathbb{C}^k(I) \times \mathbb{C}^k(I) \to \mathcal{P}\big(\mathbb{C}^k(I)\big)$ is a commutative hyperoperation and we obtain without any effort the following assertion:

3.1 Proposition. *Let $I \subset \mathbb{R}$ be an open interval and $\mathbb{C}^k(I)$ be the ring of functions $f: I \to \mathbb{R}$ of the class $\mathbb{C}^k$ for $k = 0, 1, \ldots, \infty$. Then the hypergroupoid $\big(\mathbb{C}^k(I), +\big)$ is a join space, i.e. a commutative join hypergroup.* □

Proof. By Lemma 1.1 the hypergroupoid $\big(\mathbb{C}^k(I), +\big)$ is evidently a commutative semihypergroup. Moreover, if for arbitrary pair of functions $f, g \in \mathbb{C}^k(I)$ we denote $h = g - f$ then we have $h \in \mathbb{C}^k(I)$ and $f + g = g$, thus again by Lemma 1.1 the semihypergroup $\big(\mathbb{C}^k(I), +\big)$ is a hypergroup. (This conclusion follows immediately also from the fact that $\mathbb{C}^k(I)$ is an additive group.) Now, let $f, g, u, v \in \mathbb{C}^k(I)$ be an arbitrary quadruple of functions. Recall that

$$f/g = \big\{h \in \mathbb{C}^k(I); f \in h + g\big\} = \big\{h \in \mathbb{C}^k(I); h + g \leq f\big\}$$

and similarly

$$u/v = \big\{w \in \mathbb{C}^k(I); w + v \leq u\big\}\,.$$

Suppose $f/g \approx u/v$, i.e. $(f/g) \cap (u/v) \neq \emptyset$, thus that for some function $h \in \mathbb{C}^k(I)$ we have $h + g \leq f$ and $h + v \leq u$. Then

$$h + g + u \leq f + u\,, \quad \text{i.e.} \quad g + u \leq f + u - h$$

and

$$f + h + v \leq f + u\,, \quad \text{i.e.} \quad f + v \leq f + u - h\,.$$

Then the function $w = f + u - h$ belongs to $\mathbb{C}^k(I)$ and

$$w \in (f + v) \cap (g + u)\,, \quad \text{i.e.} \quad (f + v) \approx (g + u)\,,$$

consequently the transposition axiom for $\big(\mathbb{C}^k(I), +\big)$ in the form

$$f, g, u, v \in \mathbb{C}^k(I),\ f/g \approx u/v \quad \text{implies} \quad (f + v) \approx (g + u)$$

is satisfied. Therefore the hypergroup $\big(\mathbb{C}^k(I), +\big)$ is a join space. □

4. Dual isomorphic actions of additive join spaces of continuous and smooth functions. In this last part of the present contribution we are going to construct actions of join spaces $\big(\mathbb{C}^k(I), +\big)$ on (semi)hypergroups $\big(\mathbb{LA}_2(I), *\big)$, $(\mathbb{S}, \bullet)$. These actions are in a certain sense "dual" to actions of the (semi)hypergroup $(\mathbb{S}, \bullet)$ on rings (or linear spaces) $\mathbb{C}^k(I)$ considered as phase spaces of constructed actions, for an open interval $I \subset \mathbb{R}$.

Let $k \in \{0, 1, \ldots, \infty\}$ and denote $\mathbb{LA}_2(I)_k = \big\{L(p, q); p, q \in \mathbb{C}^k(I)\big\}$ and similarly $\mathbb{S}(I) = \big\{T(1, p, q); p \in \mathbb{C}^k_+(I), q \in \mathbb{C}^k(I)\big\}$. We define a mapping

$$\delta_{CL}: \mathbb{LA}_2(I)_k \times \mathbb{C}^k(I) \to \mathbb{LA}_2(I)_k$$

by $\delta_{CL}\big(L(p, q), f\big) = L(p, q) \cdot L(1, f) = L(p, pf + q)$, for any operator $L(p, q) \in \mathbb{LA}_2(I)_k$ and any function $f \in \mathbb{C}^k(I)$. Then for any pair of functions $f, g \in \mathbb{LA}_2(I)_k$ we have

$$\begin{aligned}
\delta_{Cl}\big(\delta_{CL}(L(p, q), f), g\big) &= \delta_{CL}\big(L(p, pf + q), g\big) \\
&= L(p, pf + q) \cdot L(1, g) = L\big(p, p(f + g) + g\big) \\
&\in \big\{L(p, p(f + g) + q)\big\} \cup \big\{L(p, ph + q); f + g < h\big\} \\
&= \big\{L(p, ph + q); f + g \leq h\big\} = \big\{L(p, q) \cdot L(1, h); h \in (f + g)\big\} \\
&= \big\{\delta_{CL}(L(p, q), h); h \in (f + g)\big\} = \delta_{CL}\big(L(p, q), f + h\big)\,,
\end{aligned}$$

thus the mapping δ_{CL} is a transition map (a next state function) satisfying the Generalized Mixed Associativity Condition (GMAC), hence the triad $\big(\mathbb{LA}_2(I)_k, (\mathbb{C}^k(I), +), \delta_{CL}\big)$ is a multiautomaton — an action of the join space $\big(\mathbb{C}^k(I), +\big)$. Similarly, if

$$\mathbb{S}(I)_k = \big\{T(1,p,q); p,q \in \mathbb{C}^k(I)\big\}$$

then defining a mapping

$$\delta_{CS}\colon \mathbb{S}(I)_k \times \mathbb{C}^k(I) \to \mathbb{S}(I)_k$$

by

$$\begin{aligned}\delta_{CS}\big(T(1,p,q), f\big) &= T(1,p,q) \circ T(1,1,f)\\ &= T(1,p,pf+q)\end{aligned}$$

(here the operation "$\circ$" means usual composition of mappings), we get similarly as above that the mapping δ_{CS} satisfies GMAC, thus it serves as the next state function (the transition map) of the multiautomaton (multiaction) $\big(S(I)_k, (\mathbb{C}^k(I), +), \delta_{CS}\big)$.

4.1 Theorem. *Let $I \subset \mathbb{R}$ be an open interval. The mapping $\Phi_k\colon \mathbb{LA}_2(I)_k \to \mathbb{S}(I)_k$ defined by $\Phi_k\big(L(p,q)\big) = T(1,p,q), p,q \in \mathbb{C}^k(I)$ is an isomorphism of actions $\big(\mathbb{LA}_2(I)_k, \big(\mathbb{C}^k(I), +\big), \delta_{CL}\big)$, $\big(\mathbb{S}(I)_k, \big(\mathbb{C}^k(I), +\big), \delta_{CS}\big)$ for any value $k \in \{0, 1, \ldots, \infty\}$.*

Proof. Suppose $k \in \{0, 1, \ldots, \infty\}$. By Theorem 2.4 the mapping $\Phi_k\colon \mathbb{LA}_2(I)_K \to \mathbb{S}(I)_k$ defined by $\Phi_k\big(L(p,q)\big) = T(1,p,q)$ for any pair of functions $p,q \in \mathbb{C}^k(I)$ is an isomorphism of the semigroup $\big(\mathbb{LA}_2(I), \cdot\big)$ onto the semigroup $\big(\mathbb{S}(I)_k, \bullet\big)$; in particular Φ_k is a bijection of $\mathbb{LA}_2(I)_k$ onto $\mathbb{S}(I)_k$. Now suppose again, that p, q, $f \in \mathbb{C}^k(I)$. Then we have

$$\begin{aligned}\Phi_k\big(\delta_{CL}(L(p,q), f)\big) &= \Phi_k\big(L(p,q) \cdot L(1,f)\big)\\ &= \Phi_k\big(L(p,pf+q)\big) = T(1,p,pf+q)\\ &= T(1,p,q) \circ T(1,1,f) = \delta_{CS}\big(T(1,p,q), f\big)\\ &= \delta_{CS}\big(\Phi_k(L(p,q)), f\big)\,,\end{aligned}$$

hence

$$\big(\mathbb{LA}_2(I)_k, (\mathbb{C}^k(I), \delta_{CL})\big) \cong \big(\mathbb{S}(I)_k, (\mathbb{C}^k(I), +), \delta_{CS}\big)$$

and Φ_k is the corresponding isomorphism for any $k \in \{0, 1, \ldots, \infty\}$. □

4.2 Remark. There is possible to define a simple hyperoperation on the ring $\mathbb{C}(I)$ different from that used above. Let $f \in \mathbb{C}(I)$ be an arbitrary but fixed choosed function. Denote by $(f)^m$ the m-th power for $m \in \mathbb{N}$, i.e. $(F)^m(x) = [f(x)]^m$, $x \in I$ and $(f)^0(x) \equiv 1$. Now putting

$$h * g = \big\{(f)^m hg; m \in \mathbb{N}_0\big\}$$

for any pair of functions $h, g \in \mathbb{C}(I)$, we get that $\big(\mathbb{C}(I), *\big)$ is a commutative semihypergroup. Then defining a mapping $\delta\colon \mathbb{LA}_2(I) \times \mathbb{C}(I) \to \mathbb{LA}_2(I)$ by

$$\delta\big(L(p,q), g\big) = L(p,q) \cdot L(g,0) = L(pg,q)\,,$$

we obtain easily that this mapping δ satisfies GMAC, thus $\big(\mathbb{LA}_2(I), (\mathbb{C}(I), *), \delta\big)$ is a multiautomaton. Further, setting

$$\sigma\big(T(1,p,q), g\big) = T(1,p,q) \circ T(1,g,0) = T(1,pg,q)$$

we obtain for the pair of actions $\big(\mathbb{LA}_2(I), \mathbb{C}(I), \delta\big)$, $\big(\mathbb{S}(I), \mathbb{C}(I), \sigma\big)$ the isomorphism theorem similar to the above one.

Multiplication of operators from the semigroup $\mathbb{LA}_2(I)$ allow to construct other transition functions, i.e. actions of hyperstructures obtained from rings $\mathbb{C}^k(I)$. Investigation of those cases in more detail seems to be open.

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