Parallel Helices in Three-Dimensional Space

Veronika Chrastinová
Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics
e-mail: chrastinova.v@fce.vutbr.cz

1. Parallel curves in the plane. Let $\mathbf{p}(t) = (x(t), y(t))$ be a smooth curve in the plane. Two parallel curves

$$\mathbf{P}_+(t) = \mathbf{p}(t) + r\mathbf{n}(t),\ \mathbf{P}_-(t) = \mathbf{p}(t) - r\mathbf{n}(t) \tag{1}$$

at the distance r may be introduced where $\mathbf{n}(t)$ is the unit normal vector. They can be alternatively obtained as follows. Let us consider the envelope of circles with center $\mathbf{p}(t)$ and radius r. It consists of the points $\mathbf{P}$ satisfying

$$(\mathbf{p}(t) - \mathbf{P})^2 = r^2,\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}(t) - \mathbf{P})^2 = 2\frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}(\mathbf{p}(t) - \mathbf{P}) = 0.$$

The second equation implies $\mathbf{P} - \mathbf{p}(t) = c\mathbf{n}(t)$ and then the first one exactly gives formulae (1): the envelope consists of two parallel curves (1).

2. Parallel curves in the space. Let $\mathbf{p}(t) = (x(t), y(t), z(t))$ be a smooth curve. We introduce three equations

$$(\mathbf{p}(t) - \mathbf{P})^2 = r^2,\ \frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}(\mathbf{p}(t) - \mathbf{P}) = 0,\ \frac{\mathrm{d}^2\mathbf{p}(t)}{\mathrm{d}t^2}(\mathbf{p}(t) - \mathbf{P}) + \Big(\frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}\Big)^2 = 0 \tag{2}$$

for the point $\mathbf{P} = \mathbf{P}(t)$. If (3_1) is regarded as a spherical wave (with center $\mathbf{p}(t)$ and radius r), then (3_2) represents the enveloping surface (the intersection of two infinitesimaly close waves) and (3_3) the curve of foci $\mathbf{P} = \mathbf{P}(t)$ (the intersection of three close waves). The final result is independent of the choice of the parameter t. If the arclength $t = s$ is employed, then the Frenet formulae

$$\dot{\mathbf{p}} = \mathbf{t},\ \dot{\mathbf{t}} = \kappa\mathbf{n},\ \dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b},\ \dot{\mathbf{b}} = -\tau\mathbf{n} \quad (\dot{} = \mathrm{d}/\mathrm{d}s)$$

easily provide two solutions

$$\mathbf{P}_\pm(s) = \mathbf{p} + k\mathbf{n} \pm l\mathbf{b} \qquad (k = \frac{1}{\kappa},\ k^2 + l^2 = r^2). \tag{3}$$

Curves $\mathbf{P}_+(s)$, $\mathbf{P}_-(s)$ may be regarded as *parallel curves* to the curve $\mathbf{p}(s)$ in the three-dimensional space. The curves are real if $k^2 \le r^2$ and imaginary conjugate if $k^2 > r^2$.

3. A nontrivial result. *The curve* $\mathbf{P}_+$ (*equivalently* $\mathbf{P}_-$) *conversely determines the primary curve* $\mathbf{p}$. In more detail: *the curve* $\mathbf{p}$ *is conversely parallel to the curve* $\mathbf{P}_+$ (or: $\mathbf{P}_-$) *at the distance* r.
We refer to [1] for a tricky proof and to [2] for many generalizations. At this place we state an instructive direct verification.
Given a curve $\mathbf{p}(s)$ *and the parallel curve* $\mathbf{P}(s) = \mathbf{P}_+(s)$, *let us calculate two parallel curves* $\mathcal{P}_\pm$ *to the curve* $\mathbf{P}(s)$. *Our aim is to prove that either* $\mathcal{P}_+$ *or* $\mathcal{P}_-$ *coincides with the primary curve* $\mathbf{p} = \mathbf{p}(s)$.
We wish to determine solution $\mathcal{P} = \mathcal{P}(s)$ of the system

$$(\mathbf{P}(s) - \mathcal{P})^2 = r^2,\ \dot{\mathbf{P}}(s)(\mathbf{P}(s) - \mathcal{P}) = 0,\ \ddot{\mathbf{P}}(s)(\mathbf{P}(s) - \mathcal{P}) + (\dot{\mathbf{P}}(s))^2 = 0 \tag{4}$$

analogous to (2). Using the Frenet formulae and (3), it follows that

$$\dot{\mathbf{P}} = (\dot{k} - \tau l)\mathbf{n} + (\dot{l} + \tau k)\mathbf{b},$$
$$\ddot{\mathbf{P}} = -(\dot{k} - \tau l)\kappa\mathbf{t} + ((\dot{k} - \tau l)^{\cdot} - (\dot{l} + \tau k)\tau)\mathbf{n} + ((\dot{l} + \tau k)^{\cdot} + (\dot{k} + \tau l)\tau)\mathbf{b}.$$

Assuming moreover $\mathcal{P} = \mathbf{P} + a\mathbf{t} + b\mathbf{n} + c\mathbf{b}$, substitution into (4) gives the system

$$\begin{gathered} a^2 + b^2 + c^2 = r^2,\ (\dot{k} - \tau l)b + (\dot{l} + \tau k)c = 0, \\ (\dot{k} - \tau l)\kappa a - ((\dot{k} - \tau l)^{\cdot} - (\dot{l} + \tau k)\tau)b - ((\dot{l} + \tau k)^{\cdot} + (\dot{k} - \tau l)\tau)c \\ +(\dot{k} - \tau l)^2 + (\dot{l} + \tau k)^2 = 0. \end{gathered} \tag{5}$$

One can then directly verify that $a = 0$, $b = -k$, $c = -l$ is a solution. This provides the curve $\mathcal{P} = \mathbf{p}$ and the proof is done.
Another solution of (5) can be obtained by the substitution $b = uk$, $c = ul$. Then (5) turns into the system of two equations

$$\left(\frac{a}{r}\right)^2 + u^2 = 1,\ (\dot{k} - \tau l)\kappa a + ((\dot{k} - \tau l)^2 + (\dot{l} + \tau k)^2)(1 + u) = 0$$

for the unknown functions u and a, the intersection of an ellipse with a straight line. There are two intersection points. We already know the point $a = 0$, $u = -1$, the remaining solution can be easily found and may be omitted here.

4. Parallel helices. Assume κ, τ constant, hence $\mathbf{p}$ is a helix. Together with the arclength s along the curve $\mathbf{p}$, we introduce the arclength S along the parallel curve $\mathbf{P}$ ($\mathbf{P} = \mathbf{P}_+$ or $\mathbf{P}_-$). Employing (3), the Frenet frame

$$\mathbf{P}' = \mathbf{T} = \frac{1}{r}(-l\mathbf{n} + k\mathbf{b}) \qquad ('= \mathrm{d}/\mathrm{d}S,\ \mathrm{d}S/\mathrm{d}s = \tau r)$$

$$\mathbf{T}' = \frac{1}{\tau r^2}(\kappa l\mathbf{t} - \tau k\mathbf{n} - \tau l\mathbf{b}) = K\mathbf{N},\ K = \pm\frac{1}{\tau r^2}\sqrt{r^2(\kappa^2 + \tau^2) - 1},$$

$$\mathbf{B} = \beta(\alpha\mathbf{t} + k\mathbf{N} + l\mathbf{b}),\ T = \pm\frac{1}{\tau r^2}$$

can be found where

$$\alpha = \frac{\tau r^2}{\kappa l},\ \beta = \pm r\sqrt{\frac{r^2(\kappa^2 + \tau^2) - 1}{r^2\kappa^2 - 1}} = \pm\frac{r}{l}\sqrt{l^2 - \tau^2/\kappa^2}.$$

We have expressed the curvature and torsion K, T in terms of the primary curvature and torsion κ, τ and they are determined up to a sign $\pm$. If the construction is repeated, then the helices $\mathcal{P}_\pm$ parallel to $\mathbf{P}_\pm$ again have the original curvature and torsion κ, τ and this can be directly verified.

References

[1] Chrastinová, *Parallel curves in three-dimensional space*
(Sborník 5. Konference o matematice a fyzice 2007, UNOB)

[2] Chrastinová, V., Tryhuk, V., *Generalised contact transformations*
(to appear).