Existence of Positive Solutions of Discrete Delayed Equations

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We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$ where $s = -\infty$ and $q = \infty$ are admitted, too.

The topic of our study is a linear scalar discrete equation of k-th order

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n) x(n-i), \tag{1}$$

where $p_0 \colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $p_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}_+ := [0, \infty)$, $i = 1, \dots, k$, $k \ge 1$, a is an integer and $n \in \mathbb{Z}_a^\infty$. Let $\varphi \colon \mathbb{Z}_{a-k}^a \to \mathbb{R}$. Together with discrete equation (1), we consider an initial problem: determine a solution $x = x(n)$ of equation (1) satisfying the initial conditions

$$x(n) = \varphi(n),\ n \in \mathbb{Z}_{a-k}^a \tag{2}$$

with prescribed constants $\varphi(n) \in \mathbb{R}$. A solution of initial problem (1), (2) is defined as an infinite sequence of numbers $\{x^n\}_{n=-k}^\infty$ with $x^n = x(a+n)$, i.e., $\{x^{-k} = \varphi(a-k), \dots, x^0 = \varphi(a), x^1 = x(a+1), \dots, x^n = x(a+n), \dots\}$ such that for any $n \in \mathbb{Z}_a^\infty$ equality (1) holds.

Our aim is to find sufficient conditions with respect to the right-hand side of equation (1) in order to guarantee the existence of at least one initial function $\varphi^* \colon \mathbb{Z}_{a-k}^a \to (0, \infty)$ such that the solution $x^* = x^*(n)$ of the initial problem (1), (2) with $\varphi \equiv \varphi^*$ remains positive on $\mathbb{Z}_{a-k}^\infty$.

Theorem 1 *Let*

$$\sum_{i=1}^{k} p_i(n) > 0$$

for any $n \in \mathbb{Z}_{a-k}^\infty$, Then for the existence of a positive solution $x = x(n)$ of (1), the existence of a function $\nu \colon \mathbb{Z}_{a-k}^\infty \to \mathbb{R}^+ := (0, \infty)$ such that

$$\Delta \nu(n) \le -\sum_{i=0}^{k} p_i(n) \nu(n-i),$$

for $n \in \mathbb{Z}_a^{+\infty}$ is sufficient and necessary. Moreover $x(n) < \nu(n)$ holds on $\mathbb{Z}_{a-k}^\infty$.

Consider an auxiliary equation

$$\Delta w(n) = -\sum_{i=0}^{k} P_i(n) w(n-i) \tag{3}$$

where $P_0 \colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $P_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}_+$, $i = 1, \dots, k$, $k \ge 1$ under assumption $P_i(n) \le p_i(n)$, $i = 1, \dots, k$, $n \in \mathbb{Z}_a^\infty$.

Theorem 2 *Let*

$$\sum_{i=1}^{k} P_i(n) > 0$$

for any $n \in \mathbb{Z}_{a-k}^{\infty}$. *Assume that equation* (1) *admits a positive solution* $x = \mu(n)$ *on* $\mathbb{Z}_{a-k}^{\infty}$ *and*

$$P_i(n) \le p_i(n),$$

$i = 1, \dots, k$, $n \in \mathbb{Z}_a^{\infty}$. *Then the equation* (3) *has a positive solution* $w = w(n)$ *on* $\mathbb{Z}_{a-k}^{\infty}$ *and, moreover,* $w(n) < \mu(n)$.

Definition 1 *Let us define the expression* $\ln_q t$, $q \ge 1$, *by the formula* $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t \equiv t$ *where* $t > \exp_{q-2} 1$ *and* $\exp_s t = \exp\left(\exp_{s-1} t\right)$, $s \ge 1$, $\exp_0 t \equiv t$ *and* $\exp_{-1} t \equiv 0$ *(instead of the expressions* $\ln_0 t, \ln_1 t$, *we write only* t *and* $\ln t$ *in the following).*

Let $\ell \ge 0$ be a fixed integer. We define auxiliary functions

$$p_\ell(n) = \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \dots + \frac{k}{8(n\ln n \dots \ln_\ell n)^2}\right] \tag{4}$$

and

$$\nu_\ell(n) = \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \dots \ln_\ell n} \tag{5}$$

which play an important role in the investigation of positive solutions of an equation

$$\Delta x(n) = -p(n)x(n-k), \tag{6}$$

being a particular case of equation (1) (with $p_0 \equiv p_1 \equiv \dots \equiv p_{k-1} \equiv 0$ and $p_k \equiv p$). We assume that n in (4) and (5) is sufficiently large such that p_ℓ and ν_ℓ are well defined.

Lemma 1 *Let* $\ell \ge 0$ *be a fixed integer. Then the inequality*

$$\Delta\nu(n) \le -p_\ell(n)\nu(n-k)$$

has a solution $\nu \equiv \nu_\ell$ *provided* n *is sufficiently large.*

Lemma 2 *Let* $\ell \ge 0$ *be a fixed integer. Then the equation*

$$\Delta x(n) = -p_\ell(n)x(n-k)$$

has a positive solution $x = x(n) < \nu_\ell(n)$ *provided* n *is sufficiently large.*

Theorem 3 (Main result) *Let* $\ell \ge 0$ *be a fixed integer and* $0 < p(n) \le p_\ell(n)$ *for* $n \to +\infty$. *Then the equation* (6) *has a positive solution* $x = x(n) < \nu_\ell(n)$ *provided* n *is sufficiently large.*

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References

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