Existence of Positive Solutions of Discrete Delayed Equations

Jaromír Baštinec
Brno University of Technology, FEEC, Department of Mathematics
e-mail: bastinec@feec.vutbr.cz

Josef Diblík
Brno University of Technology
e-mail: diblik.j@fce.vutbr.cz, diblik@feec.vutbr.cz

Zdeněk Šmarda
Brno University of Technology, FEEC, Department of Mathematics
e-mail: smarda@feec.vutbr.cz

Abstract

In the paper we develop the method of inequalities to prove existence of positive solutions to linear difference equations with negative coefficients and with delays. Series of comparison results for solutions of this class of equations is derived. They are used to prove existence of positive solutions of a particular and very often investigated class of linear equations with only one delay. The relevant result is given in form of an inequality (with a suitable auxiliary function) for the coefficient of equation.

1 Introduction

We use the following notation: for integers s, q, $s \le q$, we define

$$\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$$

where $s = -\infty$ and $q = \infty$ are admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$, perhaps with other sub- or superscripts, we suppose $s \le q$.

The topic of our study is a linear scalar discrete equation of k-th order

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n) x(n-i), \tag{1}$$

where $p_0\colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $p_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}_+ := [0,\infty)$, $i = 1, \dots, k$, $k \ge 1$, a is an integer and $n \in \mathbb{Z}_a^\infty$. Let $\varphi\colon \mathbb{Z}_{a-k}^a \to \mathbb{R}$. Together with discrete equation (1), we consider an initial problem: determine a solution $x = x(n)$ of equation (1) satisfying the initial conditions

$$x(n) = \varphi(n),\ n \in \mathbb{Z}_{a-k}^a \tag{2}$$

with prescribed constants $\varphi(n) \in \mathbb{R}$. A solution of initial problem (1), (2) is defined as an infinite sequence of numbers $\{x^n\}_{n=-k}^\infty$ with $x^n = x(a+n)$, i.e.,

$$\{x^{-k} = \varphi(a-k), \dots, x^0 = \varphi(a), x^1 = x(a+1), \dots, x^n = x(a+n), \dots\}$$

such that for any $n \in \mathbb{Z}_a^\infty$ equality (1) holds. If it will be convenient, we denote the solution $x = x(n)$ of the initial problem (1), (2) as $x(n) = x(n; a, \varphi)$.

Our aim is to find sufficient conditions with respect to the right-hand side of equation (1) in order to guarantee the existence of at least one initial function

$$x(n) = \varphi^*(n),\ n \in \mathbb{Z}_{a-k}^{a}$$

with $\varphi^*\colon \mathbb{Z}_{a-k}^{a} \to (0,\infty)$ such that the solution $x^* = x^*(n; a, \varphi^*)$ of the initial problem (1), (2) with $\varphi \equiv \varphi^*$ remains positive on $\mathbb{Z}_{a-k}^{\infty}$.

2 Nonlinear Preliminaries

Let us consider a scalar discrete equation

$$\Delta u(n) = f(n, u(n), u(n-1), \dots, u(n-k)), \tag{3}$$

where $f\colon \mathbb{Z}_a^{+\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$ and $k \geq 1$ is an integer. Let $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$ be a given function. Together with discrete equation (3) we consider an initial problem: we are seeking for the solution $u = u(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ of (3) satisfying initial conditions

$$u(a-m) = \varphi(a-m),\ \ m = 0, 1, \dots, k. \tag{4}$$

The notion of a solution of the initial problem (3), (4) can be adapted easily from the Section 1 and therefore we not recall it. The existence and uniqueness of the solution of the initial problem (3), (4) is obvious as well. If f is continuous, then the initial problem (3), (4) depends continuously on the initial data.

Let two functions $b, c\colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ be given such that $b(n) < c(n)$ for each $n \in \mathbb{Z}_{a-k}^{\infty}$. For $n \in \mathbb{Z}_{a-k}^{\infty}$ we define sets

$$\omega(n) := \{(n,t)\colon t \in \mathbb{R}, b(n) < t < c(n)\}$$

and

$$\omega^*(n) := \{(t)\colon t \in \mathbb{R}, b(n) < t < c(n)\} = \mathrm{Pr}\,|_t\,\omega(n)$$

where the symbol "$\mathrm{Pr}\,|_t$" denotes the function of projection to the axis t.

Except this we define a set

$$\Omega := \{(n,t)\colon n \in \mathbb{Z}_{a-k}^{\infty}, (n,t) \in \omega(n)\} = \bigcup_{n \in \mathbb{Z}_{a-k}^{\infty}} \omega(n).$$

Obviously

$$\overline{\omega}(n) = \{(n,t)\colon t \in \mathbb{R}, b(n) \leq t \leq c(n)\},$$

$$\partial\omega(n) = \{(n,t)\colon t \in \mathbb{R}, (b(n)-t)(t-c(n)) = 0\} = \{(n, b(n)), (n, c(n))\},$$

$$\overline{\Omega} = \{(n,t)\colon n \in \mathbb{Z}_{a-k}^{\infty}, (n,t) \in \overline{\omega}(n)\} = \bigcup_{n \in \mathbb{Z}_{a-k}^{\infty}} \overline{\omega}(n)$$

and

$$\partial\Omega = \{(n,t)\colon n \in \mathbb{Z}_{a-k}^{\infty}, (n,t) \in \partial\omega(n)\} = \bigcup_{n \in \mathbb{Z}_{a-k}^{\infty}} \partial\omega(n).$$

We will formulate an auxiliary nonlinear result on existence of a solution $u = u(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ of (3) with the graph $(n, u(n))$, $n \in \mathbb{Z}_{a-k}^{\infty}$ remaining in Ω. It means, in other words, that under certain assumptions there exists at least one initial function φ such that

$$b(n) < \varphi(n) < c(n)$$

for $n \in \mathbb{Z}_{a-k}^{a}$ and

$$b(n) < u(n; a, \varphi) < c(n) \tag{5}$$

for every $n \in \mathbb{Z}_{a-k}^{\infty}$. It is easy to see, that from inequalities (5) we can deduce the existence of a positive solution of the equation (3) if our sufficient conditions will be valid for the choice: $b(n) \equiv 0$ and $c(n) > 0$, $n \in \mathbb{Z}_{a-k}^{\infty}$.

We divide the boundary $\partial\Omega$ into two nonempty disjoint subsets B_1 and B_2, where

$$B_1 := \{(n,t) \in \Omega,\ t = b(n)\},$$
$$B_2 := \{(n,t) \in \Omega,\ t = c(n)\}.$$

Now we are ready to formulate a nonlinear result, necessary for our investigation, concerning the existence of a solution of (3) with the graph lying in the set Ω (see [1, 2]).

Theorem 1 *Let the function $f\colon \mathbb{Z}_{a}^{\infty} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be continuous. If, moreover, inequalities*

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) < 0, \tag{6}$$
$$f(n, c(n), u_1, \dots, u_k) - c(n+1) + c(n) > 0 \tag{7}$$

hold for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $u_1 \in \omega^*(n-1), \dots, u_k \in \omega^*(n-k)$, then there exists an initial problem

$$u(a-m) = \varphi(a-m),\ m = 0, 1, \dots, k$$

with $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$, $\varphi(n) \in \omega^*(n)$, $n \in \mathbb{Z}_{a-k}^{a}$ such that the corresponding solution $u = u(n, a, \varphi)$ of equation (3) satisfies the inequalities

$$b(n) < u(n; a, \varphi) < c(n)$$

for every $n \in \mathbb{Z}_{a}^{\infty}$.

3 Results

The following theorems represent the main results of this contribution.

Theorem 2 *Let*

$$\sum_{i=1}^{k} p_i(n) > 0 \tag{8}$$

for any $n \in \mathbb{Z}_{a-k}^{\infty}$, Then for the existence of a positive solution $x = x(n)$ of (1), the existence of a function $\nu\colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}^{+} := (0, \infty)$ such that

$$\Delta\nu(n) \le -\sum_{i=0}^{k} p_i(n)\nu(n-i), \tag{9}$$

for $n \in \mathbb{Z}_{a}^{+\infty}$ is sufficient and necessary. Moreover $x(n) < \nu(n)$ holds on $\mathbb{Z}_{a-k}^{\infty}$.

Proof. Necessity. It is obvious since it is possible to put $\nu \equiv x$, where x is a positive solution of (1).

Sufficiency. We will use Theorem 1 with

$$f(n, u(n), u(n-1), \dots, u(n-k)) := -\sum_{i=0}^{k} p_i(n)u(n-i),$$
$$b(n) := 0,\ c(n) := \nu(n).$$

In such case $\omega^*(n) \equiv \{(t)\colon t \in \mathbb{R}, 0 < t < \nu(n)\}$. We verify inequalities (6), (7). With respect to (6) we have

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) = f(n, 0, u_1, \dots, u_k) = -\sum_{i=1}^{k} p_i(n) u(n-i).$$

It is easy to see that $u_i > 0$ if $u_i \in \omega^*(n-i)$, $i = 1, \dots, k$. Then (we use (8) as well)

$$f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n) \le -\sum_{i=1}^{k} p_i(n) < 0$$

and (6) holds. With respect to the inequality (7) we have

$$\begin{aligned} f(n, c(n), u_1, \dots, u_k) &- c(n+1) + c(n) \\ &= f(n, \nu(n), u_1, \dots, u_k) - \nu(n+1) + \nu(n) \\ &= -p_0(n)\nu(n) - \sum_{i=1}^{k} p_i(n) u_i - \nu(n+1) + \nu(n). \end{aligned}$$

Since $u_i \in \omega^*(n-i)$, then $u_i < \nu(n-i)$, $i = 1, \dots, k$, and due to (8), (9)

$$\begin{aligned} f(n, c(n), u_1, \dots, u_k) &- c(n+1) + c(n) \\ &> -p_0(n)\nu(n) - \sum_{i=1}^{k} p_i(n)\nu(n-i) - \nu(n+1) + \nu(n) \\ &= -\sum_{i=0}^{k} p_i(n)\nu(n-i) - \Delta\nu(n) \ge 0. \end{aligned}$$

Inequality (7) is valid. We conclude that all the assumptions of Theorem 1 are valid as well. With respect to the equation (1) (we change u with x) it means that there exists an initial function $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$, $\varphi(n) \in \omega^*(n)$, $n \in \mathbb{Z}_{a-k}^{a}$ such that $x = x(n, a, \varphi)$ satisfies the inequalities

$$0 \equiv b(n) < u(n; a, \varphi) < c(n) \equiv \nu(n) \tag{10}$$

for every $n \in \mathbb{Z}_a^{\infty}$. Inequality (10) coincides with the conclusion of Theorem 2. $\square$

For the proof of the main result we need a comparison result for the equation (1) and an equation

$$\Delta w(n) = -\sum_{i=0}^{k} P_i(n) w(n-i) \tag{11}$$

where $P_0\colon \mathbb{Z}_a^{\infty} \to \mathbb{R}$, $P_i\colon \mathbb{Z}_a^{\infty} \to \mathbb{R}_+$, $i = 1, \dots, k$, $k \ge 1$ under assumption $P_i(n) \le p_i(n)$, $i = 1, \dots, k$, $n \in \mathbb{Z}_a^{\infty}$.

Theorem 3 *Let*

$$\sum_{i=1}^{k} P_i(n) > 0 \tag{12}$$

for any $n \in \mathbb{Z}_{a-k}^{\infty}$. Assume that equation (1) admits a positive solution $x = \mu(n)$ on $\mathbb{Z}_{a-k}^{\infty}$ and

$$P_i(n) \le p_i(n), \tag{13}$$

$i = 1, \dots, k$, $n \in \mathbb{Z}_a^{\infty}$. Then the equation (11) has a positive solution $w = w(n)$ on $\mathbb{Z}_{a-k}^{\infty}$ and, moreover, $w(n) < \mu(n)$.

Proof. We will use Theorem 1 with

$$f(n,u(n),u(n-1),\dots,u(n-k)) := -\sum_{i=0}^{k} P_i(n)u(n-i),$$
$$b(n) := 0,\ c(n) := \mu(n).$$

Then $\omega^*(n) \equiv \{(t)\colon t \in \mathbb{R}, 0 < t < \mu(n)\}$. We verify inequalities (6), (7). With respect to (6) we have

$$f(n,b(n),u_1,\dots,u_k) - b(n+1) + b(n) = f(n,0,u_1,\dots,u_k) = -\sum_{i=1}^{k} P_i(n)u(n-i).$$

Since $u_i \in \omega^*(n-i)$ we have $u_i > 0$, $i = 1,\dots,k$ and (we use (12))

$$f(n,b(n),u_1,\dots,u_k) - b(n+1) + b(n) \le -\sum_{i=1}^{k} P_i(n) < 0.$$

Inequality (6) holds. With respect to the inequality (7) we have

$$\begin{aligned} f(n,c(n),u_1,\dots,u_k) &- c(n+1) + c(n) \\ &= f(n,\mu(n),u_1,\dots,u_k) - \mu(n+1) + \mu(n) \\ &= -P_0(n)\mu(n) - \sum_{i=1}^{k} P_i(n)u_i - \mu(n+1) + \mu(n). \end{aligned}$$

Since $u_i \in \omega^*(n-i)$, then $u_i < \mu(n-i)$, $i = 1,\dots,k$, and due to (12), (13)

$$\begin{aligned} f(n,c(n),u_1,\dots,u_k) &- c(n+1) + c(n) \\ &> -P_0(n)\mu(n) - \sum_{i=1}^{k} P_i(n)\mu(n-i) - \mu(n+1) + \mu(n) \\ &\ge -\sum_{i=0}^{k} p_i(n)\mu(n-i) - \Delta\mu(n) = 0. \end{aligned}$$

Inequality (7) is valid and all the assumptions of Theorem 1 are valid as well. With respect to the equation (11) (we change u with w) it means that there exists an initial function $\varphi\colon \mathbb{Z}_{a-k}^{a} \to \mathbb{R}$, $\varphi(n) \in \omega^*(n)$, $n \in \mathbb{Z}_{a-k}^{a}$ such that $w = w(n,a,\varphi)$ satisfies the inequalities

$$0 \equiv b(n) < w(n;a,\varphi) < c(n) \equiv \mu(n) \tag{14}$$

for every $n \in \mathbb{Z}_a^{\infty}$. Inequality (14) is equivalent with the conclusion of Theorem 3. $\square$

Definition 1 *Let us define the expression* $\ln_q t$, $q \ge 1$, *by the formula* $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t \equiv t$ *where* $t > \exp_{q-2} 1$ *and* $\exp_s t = \exp\left(\exp_{s-1} t\right)$, $s \ge 1$, $\exp_0 t \equiv t$ *and* $\exp_{-1} t \equiv 0$ *(instead of the expressions* $\ln_0 t, \ln_1 t$, *we write only* t *and* $\ln t$ *in the following).*

Let $\ell \ge 0$ be a fixed integer. We define auxiliary functions

$$p_\ell(n) = \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \dots + \frac{k}{8(n\ln n\dots\ln_\ell n)^2}\right] \tag{15}$$

and

$$\nu_\ell(n) = \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n\ln n\ln_2 n\dots\ln_\ell n} \tag{16}$$

which play an important role in the investigation of positive solutions of an equation

$$\Delta x(n) = -p(n)x(n-k), \tag{17}$$

being a particular case of equation (1) (with $p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$ and $p_k \equiv p$). We assume that n in (15) and (16) is sufficiently large such that p_ℓ and ν_ℓ are well defined.

Lemma 1 *Let $\ell \geq 0$ be a fixed integer. Then the inequality*

$$\Delta \nu(n) \leq -p_\ell(n)\nu(n-k) \tag{18}$$

has a solution $\nu \equiv \nu_\ell$ provided n is sufficiently large.

The following result is a consequence of Lemma 1.

Lemma 2 *Let $\ell \geq 0$ be a fixed integer. Then the equation*

$$\Delta x(n) = -p_\ell(n)x(n-k)$$

has a positive solution $x = x(n) < \nu_\ell(n)$ provided n is sufficiently large.

PROOF. Since the inequality (18) has a (positive) solution $\nu \equiv \nu_\ell$ provided n is sufficiently large then the proof is a straightforward consequence of Theorem 2 (we assume a is sufficiently large) with $p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$ and $p_k \equiv \nu_\ell$. □

Theorem 4 (Main result) *Let $\ell \geq 0$ be a fixed integer and $0 < p(n) \leq p_\ell(n)$ for $n \to +\infty$. Then the equation (17) has a positive solution $x = x(n) < \nu_\ell(n)$ provided n is sufficiently large.*

PROOF. It is a direct consequence of Theorem 3 (we assume a is sufficiently large) with $P_0 \equiv P_1 \equiv \cdots \equiv P_{k-1} \equiv 0$ and $P_k \equiv p(n)$ and Lemma 2 if we put $p_0 \equiv p_1 \equiv \cdots \equiv p_{k-1} \equiv 0$ and $p_k \equiv p_\ell(n)$ in (1). □

Acknowledgement

The first author was supported by the Grant 201/07/0145 of Czech Grant Agency (Prague) and by the Council of Czech Government MSM 00216 30529, the second author was supported by the Grant 201/07/0145 of Czech Grant Agency (Prague) and by the Council of Czech Government MSM 00216 30519, the third author was supported by the Council of Czech Government MSM 00216 30503.

References

[1] J. Baštinec, J. Diblík, B. Zhang: Existence of bounded solutions of discrete delayed equations, *Proceedings of the Sixth International Conference on Difference Equations*, CRC, Boca Raton, FL, 359–366, 2004.

[2] J. Diblík: Asymptotic behavior of solutions of discrete equations, *Funct. Differ. Equ.*, **11** (2004), 37–48.