β - hypergroupoids on Partially Ordered Sets

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Abstract. In this paper we will attend to the study of the multioperation $\circ$ on the partially ordered set M and to the multigroupoid ($\mathcal{M} = (M \leq, \circ, I)$ defined in [13]. The properties of the multigroupoid, especially associativity, are influenced by the structure of the carrier set M. We show that this is satisfied only for such M that is equal to a chain and hence we prove that ($\mathcal{M} = (M \leq, \circ, I)$ is associative only when the operation $\circ$ is single-valued. Such restriction on $\circ$ implies that the groupoid is an upper semilattice $\mathcal{S} = M, \vee, I$ where the operation of supremum $\vee = \circ$ is defined. These wil be studied in the opening part of this article where we repeat fundamental definitions and theorems from [13] without proofs. In the next part we introduce the concept of β-hypergroupoid and the concept of distinguishing of a hypergroupoid. In the second part of the article the distinguishing subsets of hypergroupoids are studed. Finally the relation among β-hypergroupoids and some distinguishing subsets is given.

Key words. Hypersemigroup, hypergroup equivalence and congruence on semihypergroup and hypergroups. β-hypergroupoid, the set of dual-atoms, distinguishing subsets of hypergroupoids.

1 Introduction

1.1 Definition *A hypergroupoid* (or *a multigroupoid*) is a pair $(M, \circ)$ where M is a nonempty set and $\circ :\ M \times M \to \mathcal{P}^*(M)$ is a binary hyperoperation called also a multioperation. ($\mathcal{P}^*(M)$ is the system of all nonempty subsets of M).
A semihypergroup is an associative hypergrupoid, i.e. hypergrupoid satisfying the equality $(a \circ b) \circ c = a \circ (b \circ c)$ for every triad a, b, $c \in M$.

1.2 Introduction We denote by $\mathcal{M}$ a partially ordered set M with the ordering $\leq$ and with the greatest element I which will be inscribed in the next part of this article with $\mathcal{M} = (M, \leq, I)$

1.3 Definition By the length of a chain consisting of $r + 1$ elements that is of the form
$x_0 \prec x_1 \prec x_2 \prec \ \dots \prec x_r \qquad [x_0,\, x_r]$
(where the notation $x_i \prec x_{i+1}$ means that the element x_i is covered by the element x_{i+1} - see [11]) we shall denote the non-negative number r. We define the length of the partial ordered set ($\mathcal{M} = (M, \leq, I)$ as
$max\{r_j \mid r_i, j \in J$ as lengths of chains in $M\}$.

(This definition is opposite to the definition of the length of ordered set in [11]). We suppose the partially ordered sets of finite length.

1.4 Definition We introduce for every element $u \in M$ a subset $\mathcal{U} \subseteq M$ as follows: $\mathcal{U} = \{u_i \mid u_i \geq u\}$. We define for arbitrary $x, y \in M$ on $\mathcal{M} = (M, \leq, I)$ the binary hyperoperation $\circ$ as follows:

$$x \circ y = \{ \min(\mathcal{X} \cap \mathcal{Y})\}.$$

We inscribe then the set $\mathcal{M}$ with such defined binary operation with $\mathcal{M} = (M \leq, \circ, I)$.

1.5 Definition - Remark We introduce the following very important concept. A subset Di of $\mathcal{M} = (M \leq, \circ, I)$ is called *dual ideal* of $\mathcal{M}$ if Di satisfies the followig condition:

For $x, y \in Di$ the relation $x \circ y \subset Di$ holds.

The subset U defined in 1.4 is the dual ideal of the element $u \in \mathcal{M} = (M \leq, \circ, I)$.

1.6 Lemma The binary hyperoperation $\circ$ of multiplication $\circ$ on $(\mathcal{M} = (M \leq, \circ, I)$ is idempotent.

1.7 Lemma The binary hyperoperation $\circ$ of multiplication on $\mathcal{M} = (M \leq, \circ, I)$ is commutative.

1.8 Theorem $\mathcal{M} = (M, \leq, \circ, I)$ is commutative hypergroupoid.

1.9 Theorem Every upper-ideal of $\mathcal{M} = (M, \leq, \circ, I)$ is identical with the dual-ideal of commutative hypergroupoid. $\mathcal{M} = (M \leq, \circ, I)$.

1.10 Definition Let $x \in M$. We define by $p(x)$ the position of x in M as the length of a chain between the element x and the greatest element I.

1.11 Definition Let $Q = \{q_j \mid j \in J\}, R = \{r_k \mid k \in K\}$ are subsets of M.
We say that the subset Q is smaller than the subset R in M when $q_j < r_k$ for all $j \in J$, $k \in K$ and we write $Q < R$
We say that the subset Q is incomparable to the subset R in M when $q_j \parallel r_k$ for all $j \in J$, $k \in K$ and we write $Q \parallel R$

1.12 Definition We call the triad of elements x, y, $z \in M$ an associative triad with respect to the operation $\circ$ when $x \circ (y \circ z) = x \circ (y \circ z)$.

1.13 Lemma Let $x, y, z \in M$ create a triad whose elements are linearly ordered. Then this triad is an associative triad with respect to the operation $\circ$.

Proof. Without loss of generality we will suppose that $x < y < z$. Then we have $(x \circ y) \circ z = y \circ z = z$ for $(x \circ y) = y$ and hence $y \circ z = z$ for $y < z$. By analogy $x \circ (y \circ z) = x \circ z = z$. Hence $(x \circ y) \circ z = x \circ (y \circ z)$.

1.14 Lemma Let $x, y \in M, x \parallel y$ with the greatest element I and $x \circ y = K \subset M$ where card$K > 1$. Then there exists a triad which does not satisfy the law of associativity in the multigroupoid $\mathcal{M} = (M, \leq, \circ, I)$.

Proof. There exist at least two different elements $u, v \in K$. We choose the following triad: x, y, u. Let us calculate : $x \circ (y \circ u) = x \circ u = u$ for $(x < u) \wedge (y < u)$ At either side $(x \circ y) \circ u = K \circ u = u \circ u \cup u \circ v$ and we see that the associativity is not satisfied.

1.15 Remark The Lemma 1.12 is visualised by means of a Figure 1.

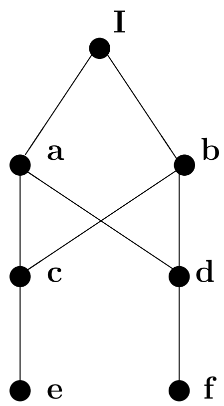

Figure 1

1.16 Theorem Let us suppose that the operation $\circ$ is single-valud for all the elements of M. Then $\circ$ is a semilattice operation of supremum and $\mathcal{M} = (M, \leq, \circ, I)$ is an upper semilattice.

Proof. The first affirmation follows from the definition of the multioperation. $\circ$ for $x \circ y = \min X \cap Y$ where X, Y are dual ideals o the elements x, y. (See 1.4, 1.5). The operation $\circ$ satisfies idempotency and commutativity (1.6, 1.7) and for the unicity of the operation the associativity to. Hence $\circ - \vee$ where $\vee$ is a semilattice operation.

2. Distinguishing Subsets on $\mathcal{M} = (M, \leq, \circ, I)$

2.1 Definition Let $N \subset M$. We say that the subset N distinguishes M if for every double $(x, y) \in M \times M$ if there exists an element $u \in M$ such that either $u \circ x \in N$ and $u \circ y \notin N$ or vice versa $u \circ x \notin N$ and $u \circ y \in N$.

2.2 Definition Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid. We say that the $\mathcal{M}$ has the property β if for every double $(x, y) \in M \times M$ for which $x \neq y$ and $x \circ y \neq I$ there exists a subset $Q \subset M$ such that $x < Q, y \parallel Q$ or vice versa $y < Q, x \parallel Q$.
.

2.3 Lemma Let $\mathcal{M} = (M, \leq, \circ, I)$ be a $\beta-$ hypergroupoid. Let x, $y \in M$, $x \neq y$, $x \circ y \neq I$. Then there exists either a subset $Q = \{q_j \mid j \in J\}$ such that $x < Q$ and $x \circ y \parallel x \circ q_j = q_j$ for $\forall j \in J$ for which $x \circ y \| x \circ q_j = q_j$ or a subset $R = \{r_k \mid k \in K\}$ such that $y < R$ and $x \parallel r_k \in R$ for which $x \circ y \parallel y \circ r_k = r_k$.

Proof. Let us permit that such a subset Q with given properties does not exist. Then for every Q for which $x < Q$, $y \| Q$ implies $Q < x \circ y$ or $Q \geq x \circ y$. The second event implies $Q \geq y$ and that is a contradiction. Hence $x < Q < x \circ y$. The hypergroupoid $\mathcal{M}$ has the property β. Hence for the double $Q, x \circ y$ where $Q < x \circ y < I$ there exists a subset $Q_0 < I$ such that $Q < Q_0 \| x \circ y$ or $x \circ y < Q_0 \| Q$. The second case implies $Q, x \circ y < Q_0$ which is a contradiction . Therefore the first situation occures. it is $Q_0 < I \wedge Q < Q_0 \| x \circ y$. Now we have $x < Q < Q_0$. From the assumption $y < Q_0$ follows $x \circ y \leq Q_0$ which is a contradiction. The dually assumption $Q_0 < y$ implies $Q_0 < x \circ y$ which is a contradiction with the property β of $\mathcal{M}$. We proved that $Q_0 \| y$ and $Q_0 \| x \circ y$ which is a contradiction with the assumption of the proof and the lemma holds true when we entertain that the second situation with the subset R is analogous to the subset Q.

2.4 Lemma Let $\mathcal{M} = (M, \leq, \circ, I)$ be a $\beta-$ hypergroupoid. Then for every double $x, y \in M \times M, x \neq y$ there exists an element $u \in M$ such that $u \circ x = I \wedge u \circ y \neq I$ or $u \circ x \neq I \wedge u \circ y = I$.

Proof. Let us suppose that x, y are dual atoms in M. The elements x, y are different. Then it is sufficient to put $u = x$ or $u = y$. We have then $u \circ x = x \circ x = x \neq I$ and $u \circ y = x \circ y = I$ or $u \circ x = y \circ x = I$ and $u \circ y = y \circ y = y \neq I$
Let $x \circ y \neq I$. For $\mathcal{M}$ has the property β there exist at least one subset $Q_0 \in M, Q_0 \neq I$ such that either $x < Q_0 \parallel y$ or $y < Q_0 \parallel x$.
Let the first situation arrive. The second is analogous. Now let $Q_0 < I$ and $y \circ Q_0 = I$. Simultaneously $x \circ Q_0 < I$ Hence Q_0 contains an element q_{j_0} for which $x \circ q_{j_0} \neq I$ and simultaneously $y \circ q_{j_0} = I$. We put $u = q_{j_0}$. Now let $y \circ Q_0 \neq I$ From the property β of $\mathcal{M}$ there exists a subset Q_1 for which $Q_0 < Q_1 \| y$ Let $Q_1 < I$ then for all elements of $Q_1 x \circ q_j < I$ for $x < Q_1$ Simultaneously either $y \circ Q_1 = I$ or $y \circ Q_1 \neq I$. In the first

case the proof is finished, the subset Q_1 contains an element q_{j_1} for which $q_{j_1} \circ y = I$ and we put $u = q_{j_1}$ In the contrary case we continue in the construction of the sequence of subsets $Q_0 < Q_1 < Q_2 < ...$ and after a finite number of steps we come to a subset Q for which $x \circ Q < I$ and $y \circ Q = I$.

2.5 Theorem Let $\mathcal{M} = (M, \leq, \circ, I)$ be a β-hypergroupoid. The one-poin subset $\{I\}$ distinguishes $\mathcal{M}$.

Proof. The proof follows from the Lemma 2.4.

2.6 Lemma Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid such that for every double $(x, y) \in M \times M, x \neq y, x \circ y < I$ there exists an element $u \in M$ such that either $u \circ x = I, u \circ y < I$ or $u \circ x < I, u \circ y = I$, it is I distinguishes $\mathcal{M}$. Then $\mathcal{M}$ has the property β.

Proof. Let us permit that $\mathcal{M}$ has not the property β. Then there exists a subset $Q_0 \subset M$ such that $x < Q_0$ and simultaneously $y < Q_0$. Hence $x \circ y = Q_0$. Then for arbitrary elements $u, v \in M$ either $u \circ x < I \wedge u \circ y < I$ or $v \circ x = I \wedge v \circ y = I$. This is a contradiction to the assumptions of the lemma and the proposition holds true.

2.7 Theorem Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid. Then the following prepositions are equivalent:

a) The one-point subset $\{I\}$ distinguishes hypergroupoid $\mathcal{M}$.

b) Hypergroupoid $\mathcal{M}$ has the property β.

Proof. (a) $\Rightarrow$(b) according to the Lemma 2.6. (b) $\Rightarrow$ (a) according to the Theorem 2.5.

2.8 Theorem Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid. Let the one-point subset $\{I\}$ distinguishes $\mathcal{M}$. Let us denote by DA the subset of dual-atoms in M. Then the subset DA distinguishes $\mathcal{M}$.

Proof. According to the presumption $\{I\}$ distinguishes $\mathcal{M}$. Hence for all doubles $(x, y) \in M \times M, x \neq y$ there exists an element $u \in M$ such that either $u \circ x = I \wedge u \circ y \neq I$ or $u \circ x \neq I \wedge u \circ y = I$.

We entertain the first situation, the second is analogous. It is $u \circ x = I$ and $u \circ y = Q \subset M - \{I\}$. Hence $Q < DA_0 \subset DA$. For all dual atoms $d \in DA_0$ holds $d \circ u \circ x = I$ for $d < I$ for arbitrary $d \in DA$. There exists a dual atom $d_0 \in DA_0$ such that $d_0 \circ u \circ y = d_0$ for $u \circ y < d_0$. We found for every double $(x, y) \in M \times M$ an element d_0 for which $d_0 \circ x \notin DA$ and $d_0 \circ y \in DA$. Hence DA distinguishes $\mathcal{M}$.

2.9 Corollary Let $\mathcal{M} = (M, \leq, \circ, I)$ be a β-hypergroupoid. Then the subset of dual-atoms distinguishes $\mathcal{M}$.

2.10 Remark Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid. From the result that the set of dual atoms DA distinguishes $\mathcal{M}$ does not follow that the one point subset $\{I\}$ distinguishes $\mathcal{M}$. We put one concrete example. Let $M = \{a, b, \alpha, \beta, I\}$ and the ordering is given by the operation $\circ$, see Figure 2.

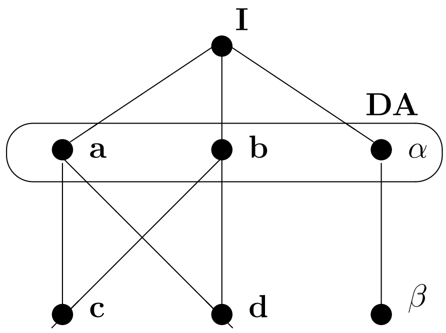

Figure 2

We see that the one-element subset $\{I\}$ does not distinguish the hypergroupoid on Figure 2. It is sufficiente to take the dpuble (c, d). Then for $u = a$ or $u = b$ or $u = c$ and or $u = d$ the multi-products $u \circ c < I, u \circ d < I$ and for the areas elements, it is for $u = I$, $u = \alpha$ and $u = \beta$ the multi-products $u \circ c, u \circ d$ are equal to I. Hence the one-point subset does not distinguish the hypergroupoid given by Figure 2.
The distinguishing of every double with respect to the set of dual atoms in hypergroupoid is given in Table 1.

	a	**b**	**c**	**d**	α	β	**i**
a	/	**a**	**c**	**d**	**a**	**a**	**a**
b	**b**	/	**c**	**d**	**b**	**b**	**b**
c	**c**	**c**	/	**c**	**a**	**a**	**a**
d	**d**	**d**	**d**	/	α	α	**d**
α	α	α	α	α	/	β	$\aleph$
β	**a**	**a**	**a**	**a**	β	/	α
I	**a**	**a**	**a**	**a**	α	α	/

Table1.

We find the distinguishing element u For wvery double of the given hypergroupoid as the intersection of the row and column. For example let us choose the double (b, c). We find as u the element c and really $c \circ c = c$ and $c \notin DA$ and $c \circ d = \{a, b\} \subset DA$, similarly for the double (b, β)we find as u the element b and $b \circ b = b \in DA$ and $b \circ \beta = I \notin DA$.

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