Mercator´s asset in mathematical cartography

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Preface

The topic of the talk is the science of map making. It is common knowledge that one cannot press the peel of an orange against a table without tearing them apart. Surprisingly, it was not until the middle of the eighteenth century that this fact was proved mathematically, and by none other than Euler. Had the earth been a cylinder, or a cone, the mapmaker´s task would have been easier - these surfaces can be flattened without shrinking. But the underlying geometry of a sphere is fundamentally different from that of a plane, consequently, one cannot create a map of the earth that faithfully reproduces all its features.

Map projections

To cope with this problem, cartographers have devised a variety of map projections - functions that assign to every point on the sphere an "image" point on the map. The choice of a particular projection depends on the intended purpose of the map - one map may show the correct distance between two points on the globe, another the relative area of countries and yet another the direction between two points.

The simplest of all projections is the cylindrical projection. Imagine the earth - represented by a spherical globe of radius R - to be wrapped in a cylinder touching it at the equator (figure 1).

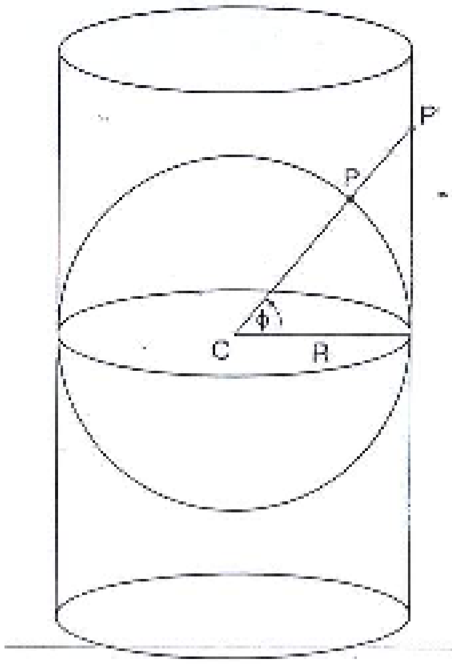

Figure 1: Cylindrical projection of the globe.

Imagine further that rays of light emanate from the center of the globe in all directions. A point P on the globe is projected onto a point P'. When the cylinder is unwrapped, we get a flat map of the entire earth.

In order to find the relation between a point P and its image P', we must first express the location of P in terms of its longitude and latitude. Denoting the longitude and latitude of P by letters λ and ϕ and coordinates of P' by x and y, we have

$$x = R\lambda,\ y = R\,\mathrm{tg}\,\phi.$$

The most striking feature of the cylindrical projection is the excessive north - south stretching at high latitudes, resulting in a drastic distortion of the shape of continents (figure 2).

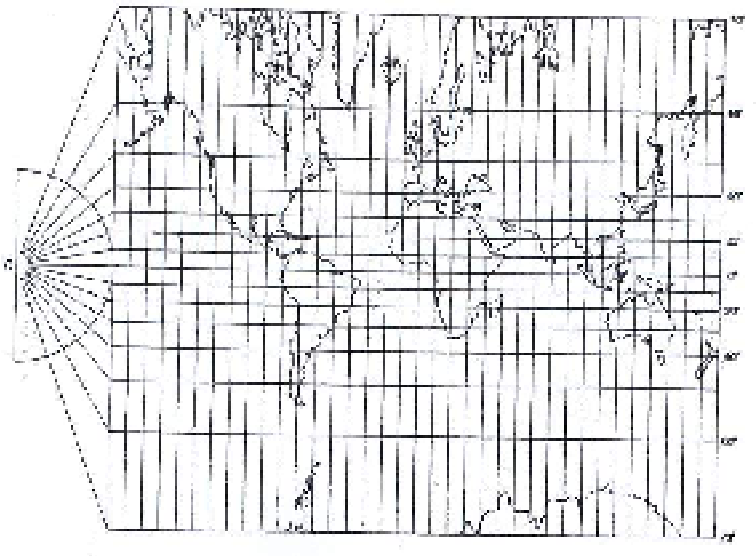

Figure 2: World map on a cylindrical projection.

The cylindrical projection has often been confused with Mercator´s projection, which it resembles superficially. However, except for the fact that both use a rectangular grid, the two projections are based on entirely different principles, as we will shortly see.

A second projection is the stereographic projection. We place the globe on a flat sheet of paper, touching it at the south pole S (figure 3). We now connect every point P on the globe by a straight line to the north pole N and extend this line until it meets the plane of the map at the point P'. The equator goes over to a circle e, which we may think of as the unit circle.

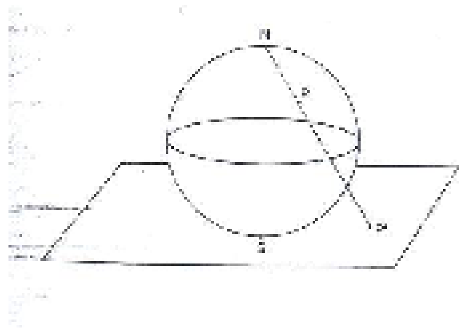

Figure 3: Stereographic projection from the north pole.

Let the globe have a unit diameter. This will ensure that circle e has a unit radius. Consider now a point P with latitude ϕ on the globe. We wish to determine the location of its image P' on the map.

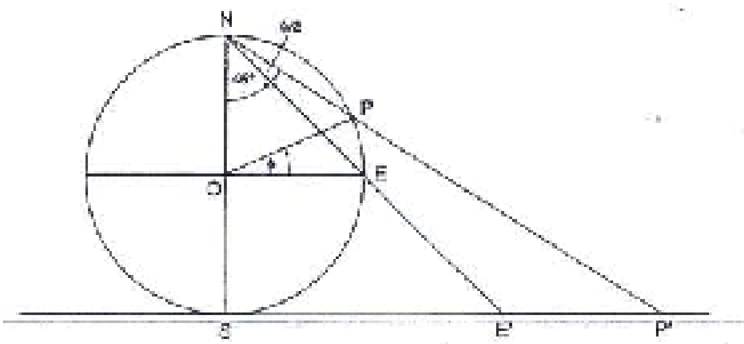

Figure 4: Geometry of the stereographic projection.

Figure 4 shows a cross section of the globe, with E representing a point on the equator. We have $|SN| = 1$, $|\measuredangle ONE| = 45°$, $|\measuredangle EOP| = \phi$ and $|\measuredangle ENP| = \frac{\phi}{2}$. Therefore, $|\measuredangle ONP| = (45° + \frac{\phi}{2})$ and thus P' is located at a distance

$$|SP'| = \mathrm{tg}(45° + \frac{\phi}{2})$$

from the south pole on the map.

Equation leads to an interesting result. Let P and Q be two points with the same longitude but opposite latitude on the globe. We get

$$|SQ'| = \mathrm{tg}(45° - \frac{\phi}{2}) = \frac{1 - \mathrm{tg}\frac{\phi}{2}}{1 + \mathrm{tg}\frac{\phi}{2}} = \frac{1}{\mathrm{tg}(45° + \frac{\phi}{2})} = \frac{1}{|SP'|}.$$

Thus $|SP'| \cdot |SQ'| = 1$. Two points in the plane fulfilling this condition are said to be inverse points with respect to the unit circle. From this one can show that the stereographic projection is direction - preserving, or conformal.

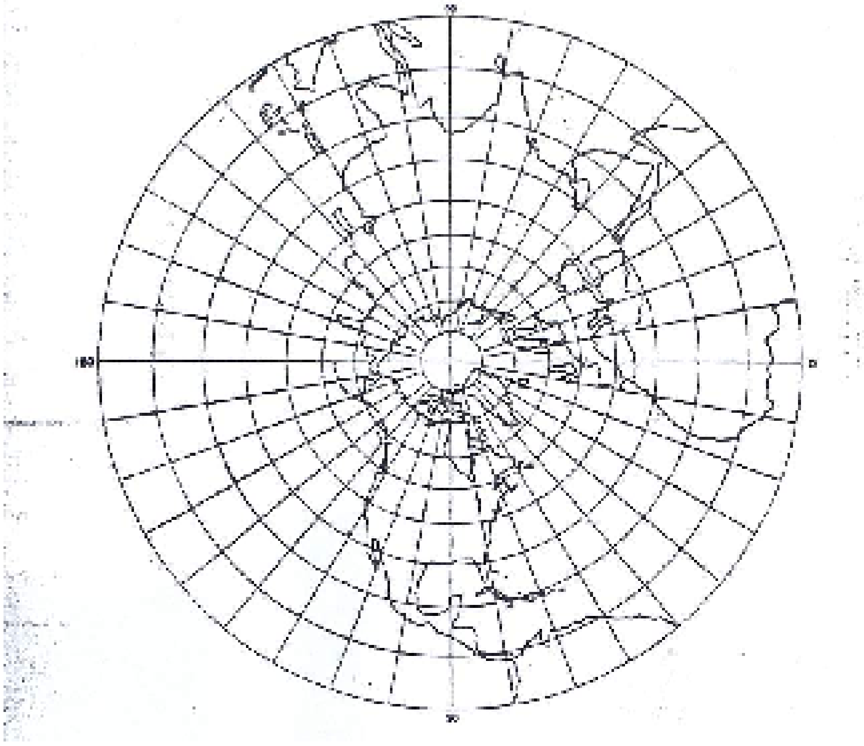

Figure 5: The northern hemisphere on a stereographic projection.

Figure 5 shows the northern hemisphere on a stereographic projection in which the globe touches the map at the north pole.

Imagine you are the navigator of a boat about to leave port headed in a certain direction. You set your compass at your chosen bearing and then follow that bearing steadfastly. For many years it had been believed that a path of constant bearing - known as a loxodrome - is an arc of a great circle (fig. 6).

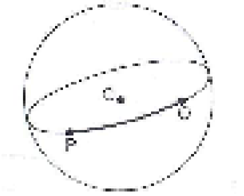

Figure 6: An arc of a great circle.

But the Portuguese Pedro Nunes in the sixteenth century showed that loxodrome is actually a spiral curve that gets ever closer to either pole, winding around it indefinitely but never reaching it. (fig. 7).

Figure 7: Loxodrome.

The challenge that faced cartographers in the sixteenth century was to design a map that would show all loxodromes as straight lines. It befell a Flemish mapmaker to come to the mariners rescue.

Gerardus Mercator

Figure 8: Gerardus Mercator.

Gerardus Mercator, by general consensus the famous mapmaker in history, was born in Flanders (now Belgium but then part of Holland) on March 5, 1512. He entered the University of Louvain in 1530 and soon after graduating established himself as one of Europe´s leading mapmakers.

Mercator´s promising career was threatened in 1544 when he was arrested as a heretic for practising protestantism in a catholic country. He barely saved his life and subsequently fled to neighboring Duisburg (now

Germany), where he settled in 1552. He remained there for the rest of his life.

Before Mercator, mapmakers decorated their charts with fanciful mythological figures and imaginary lands of their own creation. Mercator was the first to base his maps entirely on the most recent data collected by explorers, and in so doing he transformed cartography from an art to a science.

It was in 1568 that Mercator set himself the task of inventing a new map projection. From the outset he was guided by two principles: the map was to be laid out on a rectangular grid, with all circles of latitude represented by horizontal lines parallel to the equator and equal to it length, and all meridians showing as vertical lines perpendicular to the equator. And the map would be conformal, for only such a map could preserve the true direction between any two points on the globe.

Now on the globe, the circles of the latitude decrease in size as their latitude increases. But on Mercator´s map these same circles show as horizontal lines of equal length. Consequently, each parallel on the map is stretched horizontally by a factor that depends on the latitude of that parallel.

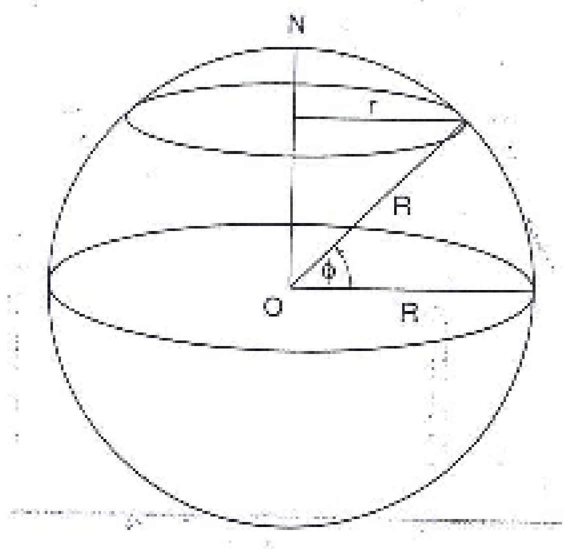

Figure 9: Circle of latitude ϕ on the globe.

Figure 9 shows a circle of latitude ϕ. Its circumference is $2\pi r = 2\pi R\cos\phi$ on the globe, whereas on the map its length is $2\pi R$. It is thus stretched by a factor $\frac{2\pi R}{2\pi R\cos\phi} = \sec\phi$.

And now Mercator was ready to produce his trump card: in order for the map to be conformal, the east - west stretching of the parallels must be accompanied by an equal north - south stretching of the spacing between the parallels, and this north - south stretching progressively increases as one goes to higher latitudes (fig. 10). This is the key principle behind his map.

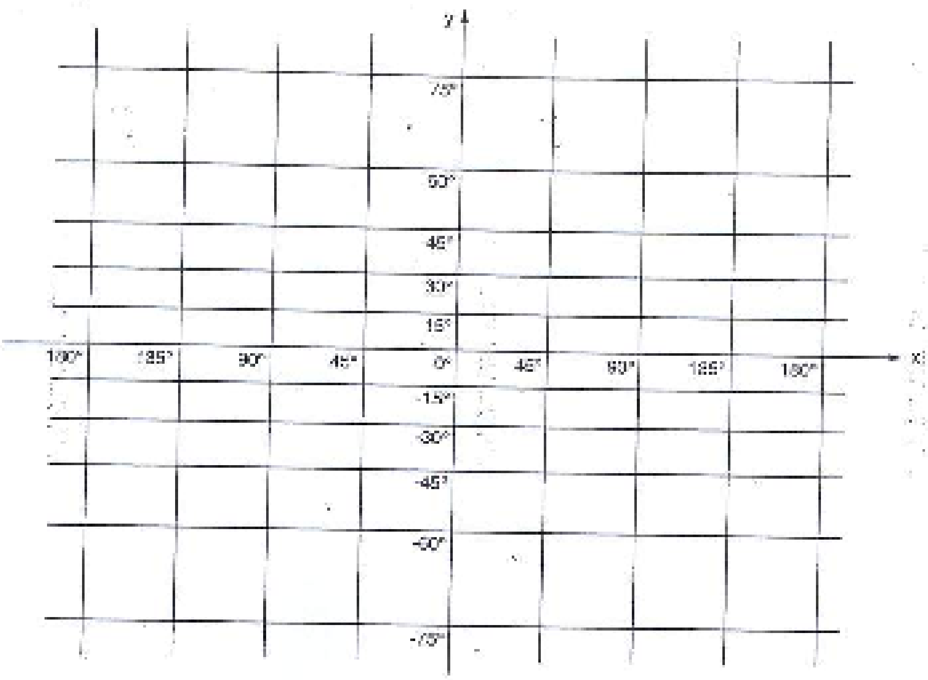

Figure 10: The Mercator grid.

Exactly how Mercator did the spacing between successive parallels is not known and is still being debated by historians of cartography. He left no written record of his method.

Mercator died in Duisburg on December 2, 1594. Yet his most famous achievement, the map that bears his name, was not immediately embraced by the maritime community. The fact that Mercator had not given a full account of how he had progressively increased the distance between the parallels only added to the confusion.

Edward Wright

It was left to Edward Wright (1560 - 1615), an English mathematician, to give the first accurate account of the principles underlying Mercator´s map. Wright used numerical integration to evaluate $\int_0^\phi \sec\phi \, d\phi$. Let us follow his plan.

Figure 11 shows a small spherical rectangle defined by the circles of longitudes λ and $\lambda + \Delta\lambda$ and circles of latitudes ϕ a $\phi + \Delta\phi$. Let a point $P(\lambda, \phi)$ on the sphere go over to the point $P'(x, y)$ on the map. Then the spherical rectangle will be mapped onto a planar rectangle defined by the lines x, $x + \Delta x$ and y, $y + \Delta y$, where $\Delta x = R\Delta\lambda$ (fig. 12). The requirement that the map be conformal means that these two rectangulars must be similar! Thus we are led to the equation

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{R\Delta\lambda} = \frac{R\Delta\phi}{R\cos\phi\Delta\lambda},$$

or

$$\Delta y = (R\sec\phi)\Delta\phi.$$

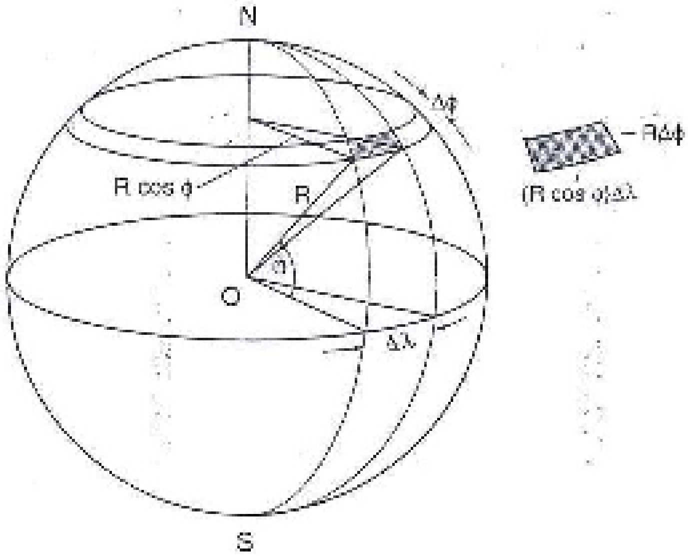

Figure 11: Spherical rectangle on the globe.

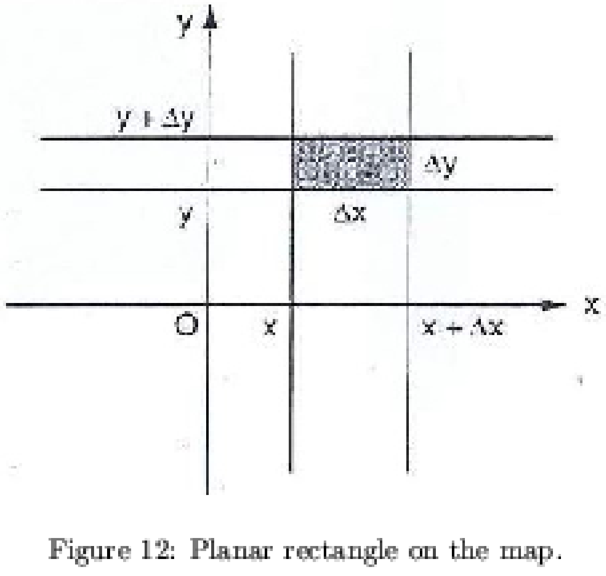

Figure 12: Planar rectangle on the map.

Equation is a finite difference equation. Nowadays, of course, we would write equation as a differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = R \sec\phi,$$

whose solution is

$$y = R\int_0^{\phi} \sec t \,\mathrm{d}t.$$

Epilogue

We are now in a position to write down the coordinates (x, y) of a point P' on Mercator´s map in terms of longitude λ and latitude ϕ of corresponding point P on the globe. The difference equation $\Delta x = R\Delta\lambda$ has the obvious solution $x = R\lambda$. We thus have

$$x = R\lambda,\ y = R\int_0^{\phi} \sec t \,\mathrm{d}t.$$

Figure 13 shows the world as it appears on Mercator´s map. Because of the excessive north - south stretching at high latitudes, the map is confined to latitudes from 75° north to 60° south.

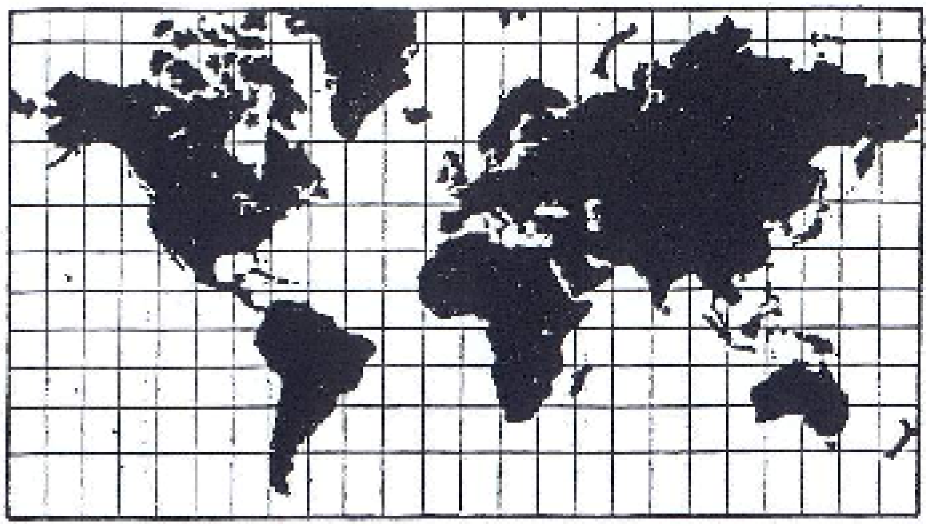

Figure 13: The world on Mercator´s map.

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