An Influence of Parameters on the Shape of NURBS Surfaces

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Abstract

NURBS (Non-Uniform Rational B-Spline) surfaces became foundation-stone in many CAD systems. Its biggest advantage is unlimited design possibilities with local control. In this paper, it will be shown the geometric essence of the knot vector changes and the influence of the different types of knot vector on the shape of the result surface. We also show the influence of the weight and the position of the control point.

1 Introduction

Our intention here is to highlight the influence of different changes on the shape of NURBS surface – position and weight of control point, knot vector. NURBS surfaces are implemented in many graphic software (e.g. Maya, FormZ, Blender). Users can modify free-form shaped surfaces with a position of the control points, its tangents and also with the knot vectors. In this article we report on the geometric essence of it. Base functions will be drawn with Maple and result surfaces will be drawn with FemDev – the testing environment for RFEM 3D.

1.1 Related work

The first works were published in seventies by founders of B-spline – Ch. de Boor, Iso Schoenberg (1). The theoretical base of NURBS is the book by Peigl, Tiller (2) and Shene (3). This book also contains the algorithms of computing NURBS and describes several properties of NURBS. Modification of weighs is described in (4).

In (5) is studied the sensitivity of a spline function, represented in terms of B-splines, to perturbations of the knots. The effect of the modification of knot values on the shape of B-spline curve is examined in (6). Using of genetic algorithms for knot vectors optimization is interesting part – see (8). Also (7) try to optimize knot vector by setting these knot values automatically, taking into account some good measures of the shape.

1.2 Overview

NURBS surfaces are defined recursively by means of the B-spline functions. This is presented in Section 2. In Section 3 is shown the influence of the position and the weight of the control points to the NURBS surface. Section 4 briefly outlines the influence of two types of the knot vector (uniform, non-uniform) on the shape of the NURBS surfaces. It contains simple examples to explain the general relations.

2 B-Spline, NURBS surface

Definition 1. Knot vector t is a non-decreasing sequence of the positive real numbers $\mathbf{t} = (t_0, t_1, \ldots t_n)$. The knot vector is called **uniform**, when $t_{i+1} - t_i = t_i - t_{i-1}$ for $i = 1, 2, \ldots, n-1$, else the knot vector is called **non-uniform.**

Definition 2. Let $\mathbf{t} = (t_0, t_1, \ldots t_n)$ be a knot vector. **B-spline function** N_i^k of degree k is defined by

$$N_i^0(t) = \begin{cases} 1 & \text{for} \quad t \in \langle t_i, t_i + 1) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$N_i^k(t) = \frac{t - t_i}{t_{i+k} - t_i} N_i^{k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1}^{k-1}(t), \tag{2}$$

where $0 \leq i \leq n - k - 1, 1 \leq k \leq n - 1, \frac{0}{0} := 0$. B-Spline function is often called **base function**

for NURBS curves and surfaces.

Definition 3. Let have:
net $(q+1) \times (r+1)n$ of control points P_{ij}, where $i = 0, \ldots, q, n = 0, \ldots, r$,
$(q+1)(r+1)$ positive real numbers $w_{i,j}$ called weights,
the degree of the surface of the columns n and the degree of the surface for the rows m,
the column knot vector $\mathbf{u} = (u_0, u_1, \ldots t_{n+q+1})$,
the row knot vector $\mathbf{v} = (v_0, v_1, \ldots t_{m+r+1})$,
than NURBS surface is defined by

$$C(u, v) = \frac{\sum_{i=0}^{q} \sum_{j=0}^{r} w_{ij} P_{ij} N_i^n(u) N_j^m(v)}{\sum_{i=0}^{q} \sum_{j=0}^{r} N_i^n(u) N_j^m(v)}. \tag{3}$$

If

$$R_i^j(u, v) = \frac{\sum_{i=0}^{q} \sum_{j=0}^{r} N_i^n(u) N_j^m(v)}{\sum_{i=0}^{q} \sum_{j=0}^{r} N_i^n(u) N_j^m(v)},$$

than the eq. 3 can be written as

$$C(u, v) = \sum_{i=0}^{q} \sum_{j=0}^{r} R_i^j(u, v) w_{ij} P_{ij}, \tag{4}$$

where $(u, v) \in \langle u_0, u_{n+q+1}) \times \langle v_0, v_{n+r+1})$.

Definition 4. Let $C(t)$ be a NURBS curve defined in Section 2. The value $\bar{t}$ for parameter t has multiplicity s, when $\bar{t} = t_i, \bar{t} = t_{i+1}, \ldots, t_{i+s-1}$ (where $i = 0, \ldots, m + n + 1 - s$).

Lemma 2.1. *Let m, n be the degrees of $S(u, v)$. Then $S(u, v)$ is C_{n-s} (resp. C_{n-w}) continuous at the point $S(\overline{u}, v)$ (resp. $S(u, \overline{v})$ for the parameter $u = \overline{u}$ (resp. $v = \overline{v}$) where $\overline{u}$ (resp. $\overline{v}$) is of multiplicity s (resp. w).*

The proof is trivial and we omit it.

Theorem 2.2. *$N_i^m(u) N_1^m(v)$ is zero if parameters (u, v) are outside of rectangle $\langle u_i \times u_{i+n+1}) \times \langle v_j \times u_{j+m+1})$.*

The proof is in Shene (3).

3 The influence of the position and the weight of the control points

The most useful is changing of the control points. User can modify the final surface with drag and drop. The adventage is that this modification is local. The point P_{ij} changes the surface for the intervals $\langle u_i, u_{i+n+1}) \times \langle v_i, v_{i+m+1})$, where m, n are the surface degrees.

Let $S(u, v)$ be a NURBS surface given by:

Degree: $m = n = 2$
Knot vectors: $u = (0, 0, 0, 0.5, 1, 1, 1)$ $v = (0, 0, 0, 0.5, 1, 1, 1)$
Control points:
1. column: $(-1, 0, 6), (0, -2, 6), (0, -4, 6), (-1, -8, 6)$
2. column: $(0, 0, 0), (0, -2, 0), (0, -4, 0), (0, -8, 0)$
3. column: $(6, 0, 0), (6, -2, 0), (6, -4, 0), (6, -8, 0)$
4. column: $(8, 0, 6), (6, -2, 6), (7, -4, 6), (8, -8, 6)$

Weights: $w_{ij} = 1$ for all $P_{ij}, i = 0, \ldots, n, j = 0, \ldots, m$.

On fig. 1 we change the position of point P_{11} on coordinates $(-5, -2, 0)$. The continuity is not affected.

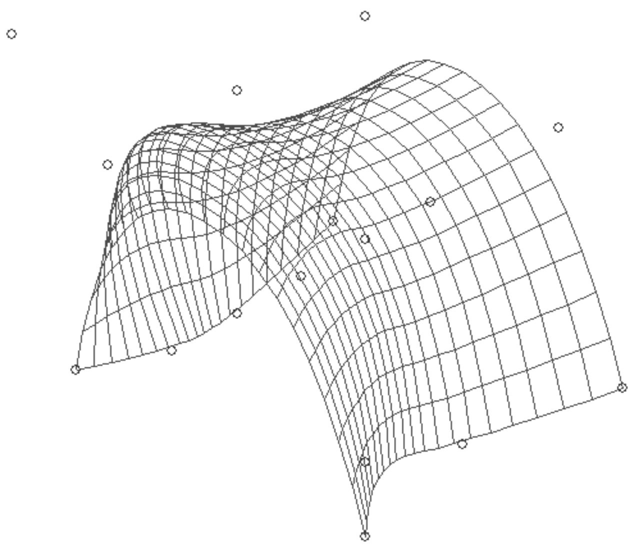

Figure 1: Influence of the control point position (FemDev)

Every control point on NURBS surface has its weight. This number shows, how powerful the point will be. Implicitly, all weight are equal to one. For weights smaller than one has control point less power. For weight bigger than one, the power grows up.

Piegl and Tiller in (2), (4) show the opportunity of negative weights to construct arcs. But it is not user friendly and it can be done with positive numbers as well. Fig. 2 shows the final NURBS surfaces for different weights ($w_i = 4.0$, $w_i = 0.2$).

4 The influence of the knot vectors

Base functions for NURBS surfaces are products $N_i^n N_j^m$ for $i = 0, \ldots, q$ and $j = 0, \ldots, r$.

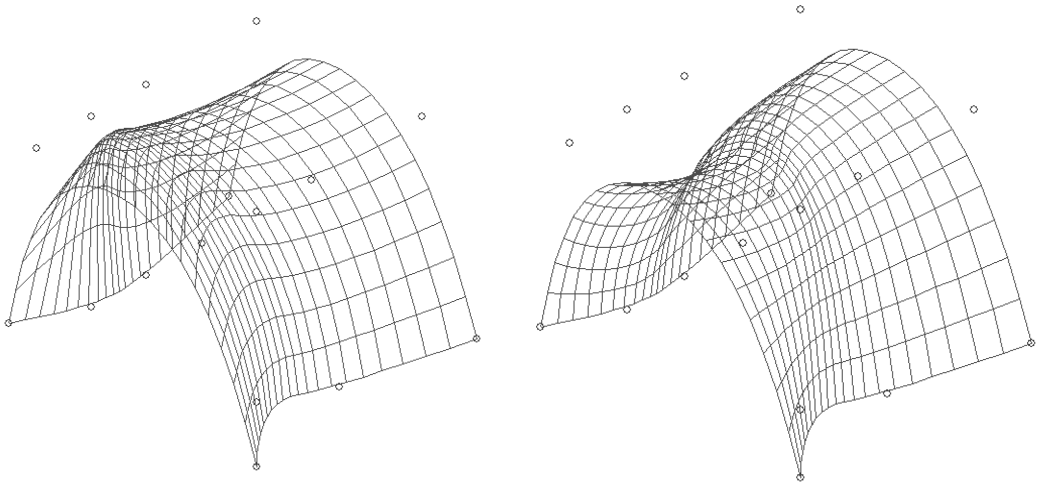

Figure 2: The influence of the weights on the shape of the NURBS surface, $w_i = 4.0$, $w_i = 0.2$, (FemDev)

$$\sum_{i=0}^{q}\sum_{j=0}^{r} N_i^n N_j^m = N_0^n N_0^m + N_0^n N_1^m + \ldots + N_0^n N_r^m + N_1^n N_0^m + \ldots + N_q^n N_r^m \tag{5}$$

Every point P_{ij} has the different base function $N_i^n(u)N_j^m(v)$ for parameters (u, v). The value of the base function indicates percentage influence of this point to the surface. This number is generally in the interval $\langle 0, 1\rangle$ because of B-Spline properties - nonnegativity and partition of unity (see (3)).

4.1 Uniform knot vector

Let $\mathbf{u}, \mathbf{v}$ be the uniform knot vectors
$u = (0, 1, 2, 3, 4, 5, 6, 7)$
$v = (0, 1, 2, 3, 4, 5, 6, 7)$.

We choose $(u, v) = (2.8, 4.2)$.
The value 2.8 lies in the interval $\langle 2, 3)$ and the value $v = 4.2$ in the interval $\langle 4, 5)$. So, N_0^2, N_1^2, N_2^2 are non-negative for parameter u and for parameter v multinomials N_2^2, N_3^2, N_4^2 (because of Theorem 2.2).
Obviously we have

$$\begin{aligned} S(2.8, 4.2) &= P_{02}N_0^2(2.8)N_2^2(4.2) + P_{03}N_0^2(2.8)N_3^2(4.2) + P_{04}N_0^2(2.8)N_4^2(4.2) \\ &+ P_{12}N_1^2(2.8)N_2^2(4.2) + P_{13}N_1^2(2.8)N_3^2(4.2) + P_{14}N_1^2(2.8)N_4^2(4.2) \\ &+ P_{22}N_2^2(2.8)N_2^2(4.2) + P_{23}N_2^2(2.8)N_3^2(4.2) + P_{24}N_2^2(2.8)N_4^2(4.2), \end{aligned}$$

and we obtain

$$\begin{aligned} S(2.8, 4.2) &= P_{02}0.0064 + P_{03}0.0132 + P_{04}0.0004 \\ &+ P_{12}0.2112 + P_{13}0.4356 + P_{11}0.0132 \\ &+ P_{22}0.1024 + P_{23}0.2112 + P_{21}0.0064. \end{aligned}$$

For example, point P_{13} has in time $(2.8, 4.2)$ influence 43.56 percent on the result surface.

On fig. 4.1 the base functions for this vector and result NURBS surface (5×5 control points, degree 2, all weights equal to 1) are drawn.

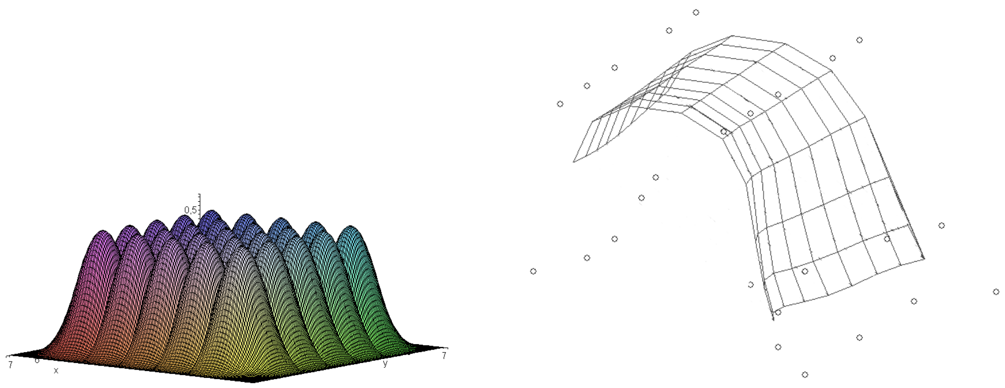

Figure 3: Uniform knot vector – base functions

4.2 Non-uniform knot vector

Let $\mathbf{u}, \mathbf{v}$ be non-uniform knot vectors
$u = (0, 0, 0, 1/3, 2/3, 1, 1, 1)$
$v = (0, 0, 0, 1/3, 2/3, 1, 1, 1)$.

For $(u, v) = (0, 0)$ are $N_0^2(u) = 1, N_0^2(v) = 1$. Other base polynomials are equal to zero. As a consequence, there is only one non-zero base function:

$$N_0^2(u)N_0^2(v) = 1.$$

Therefore, point P_{00} lies on the surface, because base function has influence 100% here.

The situation is same for parametres $(u, v) = (0, 1), (1, 0), (1, 1)$. Non-zero base functions are:

$$N_0^2(u)N_4^2(u) = 1,$$

$$N_4^2(u)N_0^2(u) = 1,$$

$$N_4^2(u)N_4^2(u) = 1.$$

So, points P_{04}, P_{40}, P_{44} lie on the result NURBS surface.

Generally, if the knot vector is:

$$\underbrace{0, 0, \ldots, 0}_{degree+1}, \ldots, \underbrace{1, 1, \ldots, 1}_{degree+1},$$

than the NURBS surface $m \times n$ points passes through the points $P_{00}, P_{0n}, P_{m0}, P_{mn}$. The proof can be done by generalization of the previous computation.

Base functions and the result surface are drawn on Fig. 4.

Let $\mathbf{u}, \mathbf{v}$ be non-uniform knot vectors
$u = (0, 0, 0, 0.5, 0.5, 1, 1, 1)$
$v = (0, 0, 0, 0.5, 0.5, 1, 1, 1)$.

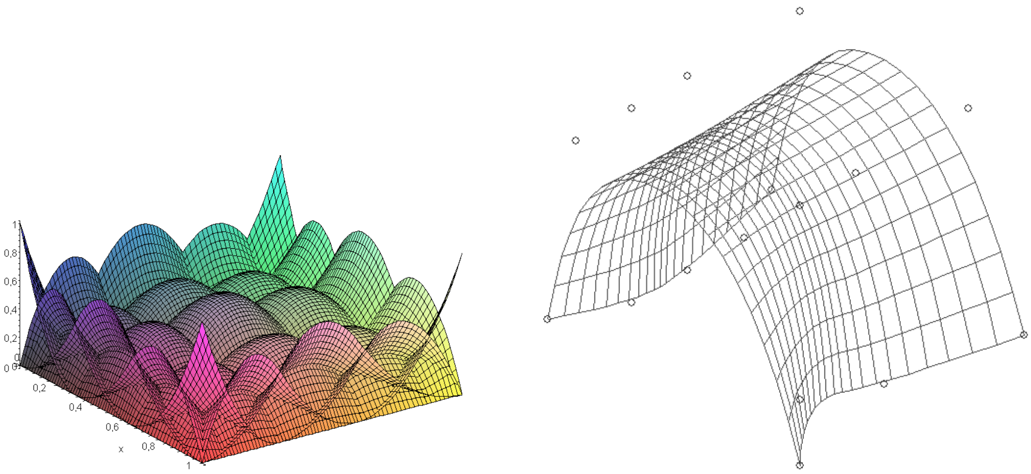

Figure 4: Non-uniform knot vectors $(0, 0, 0, 1/3, 2/3, 1, 1, 1)$

Because of Lemma 2.1, the continuity in the points $S(u, v)$ of surface is C_0 for parameters $u, v = 0.5$. As you can see on fig. 5, points $P_{02}, P_{20}, P_{22}, P_{24}, P_{42}$ lie on the NURBS surface. Base functions for this points are equal to one. The base functions are drawn with Maple on fig. 5.

$$N_0^2(u)N_2^2(u) = 1$$
$$N_2^2(u)N_0^2(u) = 1$$
$$N_2^2(u)N_2^2(u) = 1$$
$$N_2^2(u)N_4^2(u) = 1$$
$$N_4^2(u)N_2^2(u) = 1.$$

With regard to previous part, points $P_{00}, P_{40}, P_{04}, P_{44}$ also lie on the final surface.

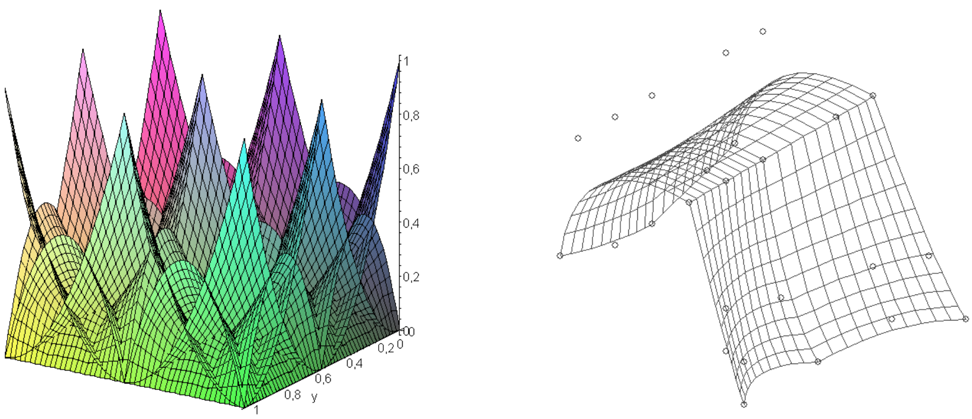

Figure 5: Non-uniform knot vectors $(0, 0, 0, 0.5, 0.5, 1, 1, 1)$

Let $S(u, v)$ and $\overline{S}(\overline{u}, \overline{v})$ be two NURBS surfaces with non-uniform knot vectors
$u = (0, 0, 0, 0.2, 0.4, 1, 1, 1)$
$v = (0, 0, 0, 0.2, 0.4, 1, 1, 1)$.

$\overline{u} = (0, 0, 0, 0.6, 0.8, 1, 1, 1)$
$\overline{v} = (0, 0, 0, 0.6, 0.8, 1, 1, 1)$.

What are the differences between these two surfaces? For example, let $(u, v) = (0.7, 0.7)$. We compute the values of the base functions and compare them.

Surface $S(u, v)$

$$0.7 \in \langle 0.4, 1) \Longrightarrow N_2^2(0.7), N_3^2(0.7), N_4^2(0.7) \neq 0 \Longrightarrow$$

$$\text{points } P_{ij} \text{ have the influence on the result surface, where } i, j = 2, 3, 4$$

Surface $\overline{S}(u, v)$

$$0.7 \in \langle 0.6, 0.8) \Longrightarrow N_1^2(0.7), N_2^2(0.7), N_3^2(0.7) \neq 0 \Longrightarrow$$

$$\text{points } P_{ij} \text{ have the influence on the result surface, where } i, j = 1, 2, 3$$

It is clear, that control points for parameters $(0.7, 0.7)$ are different. For example, point P_{22} has influence 3.5 percent on surface S, but on surface $\overline{S}$ has influence 66 percent.

This is clearly shown on Figure 4.2, where are base functions for both surfaces. On fig. 4.2 are drawn result surfaces.

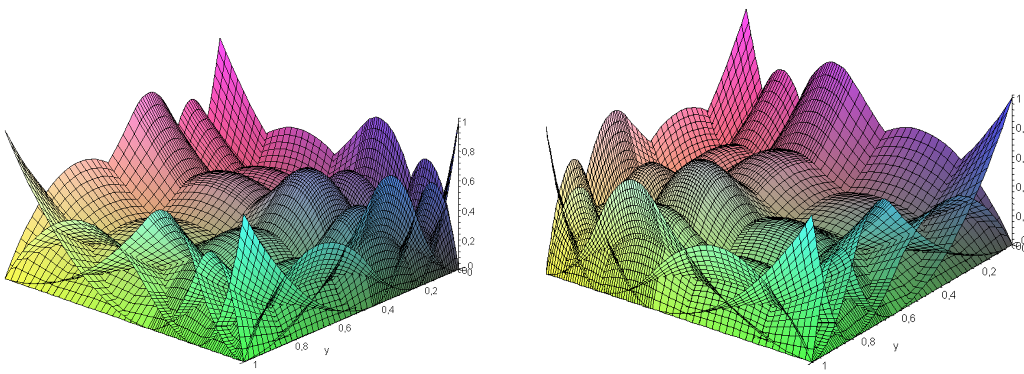

Figure 6: Comparing of the base functions for non-uniform knot vectors

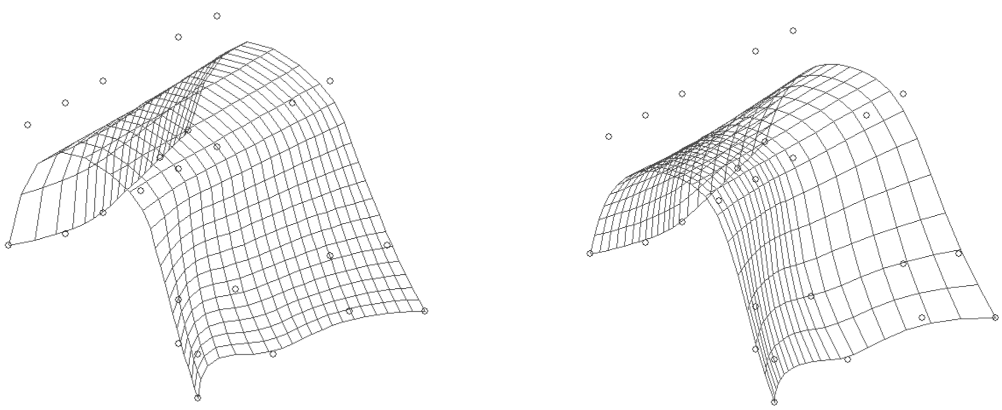

Figure 7: Result surfaces

5 Conclusion

The discussion in this article has given an overview of the knot vectors using for NURBS surfaces. We tried to en-light the geometric essence of knot vectors changes. The main types of the knot vectors and results surfaces were shown. The influence of the control points was also discussed.

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