COVARIANT ASPECTS OF STRUCTURES OF SOLUTION SPACES

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I. Motivation.

On two examples we would like to demonstrate how a structure of solutions of certain equations can be described by using transformations, morphisms of those equations into special, canonical objects of the corresponding Brandt groupoid.

II. Abel functional equation.

Consider Abel functional equation

$$f(\varphi(x)) = f(x) + 1, \tag{1}$$

where φ is a given continuous strictly increasing real-valued function defined on a half-open interval $[a, b) \subseteq \mathbb{R}, b \leq \infty$, mapping it onto a half-open interval $[c, b)$, and $\varphi(x) > x$ for all $t \in [a, b)$. On the basis of the results of B. Choczewski [**3**] and E. Barvínek [**1**], see also M. Kuczma, B. Choczewski, R. Ger [**4**], there exists an invertible solution f_0. If we define $g(t) := f(f_0^{-1}(t))$, we get

$$g(t) + 1 = g(t + 1) \tag{2}$$

having the general solution $g(t) = t + P(t)$, P being a periodic function with the period $1: \quad P(t + 1) = P(t)$. Summarizing these relations, we have $t + P(t) = f(f_0^{-1}(t))$, hence $f(x) = f_0(x) + P(f_0(x))$. Thus the general solution f of Abel functional equation (1) is

$$f(x) = f_0(x) + P(f_0(x)), \tag{3}$$

where $P(t + 1) = P(t)$ on $[f_0(a), \infty)]$.

Remark 1. Let us emphasize that we get the structure of all solutions of (1) by transforming this equation into its *canonical* Abel equation (2), solving it and then transforming such obtained general solution of (2) back into solutions of the original equation (1).

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III. Differential equations.

For the demonstration of our approach, consider for simplicity the second order linear homogeneous differential equation in the Jacobi form

$$y'' = p(x)y, \tag{p}$$

on an interval $I \subset \mathbb{R}, p \in C^0(I)$. The general solution of (p) is two-dimensional vector space $y = c_1 y_1 + c_2 y_2$, where y_1, y_2 are two linearly independent solutions of (p) of class $C^2(I)$ with non-vanishing Wronskian. Another copy of such an equation let us denote by

$$z'' = q(t), \quad q \in J \subset \mathbb{R} \tag{q}$$

with the solution space $c_1 z_1 + c_2 z_2$. Due to E. E. Kummer [**5**] and O. Borůvka [**2**], the most general pointwise global transformation of (p) into (q) is of the form

$$y(x) = f(x).z(h(x)), \quad x \in I, \quad h \in C^3(I), \quad h(I) = J, \quad f = |h'|^{-1/2}.$$

Again according to [**2**], we may specify (q) as the *canonical* equation

$$u'' = -u \tag{4}$$

with the general solution $u = c_1 \sin t + c_2 \cos t$. Hence the general solution of the original equation (p) is

$$y(x) = f(x).[c_1 \sin h(x) + c_2 \cos h(x)],$$

where $f(x) = |h'(x)|^{-1/2}$ due to the Jacobi form of equation (p).

Remark 2. Again we have seen that the structure of all solutions of equation (p) can be obtained by transforming this equation into its *canonical* equation (4), by solving it and then transforming such obtained general solution back into solutions of the original equation.

IV. Comments.

The same approach can be applied in other situations, e.g. for functional differential equations, see [**6**].

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