A NOTE ON PARACOMPACTNESS AND FULL NORMALITY WITHOUT WEAK SEPARATION AXIOMS

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ABSTRACT. In this paper we develop a general method which allows to omit T_0 and T_1 axioms in some covering theorems concerning paracompactness and full normality.

1. Introduction

Throughout this paper, by a space we always mean a topological space. In a space X a point $x \in X$ is in the *θ-closure* of a set $A \subseteq X$ and we write $x \in \mathrm{cl}_\theta A$ if every closed neighbourhood of x intersects A. A filter base Φ in X has a *θ-cluster point* $x \in X$ if $x \in \bigcap \{\mathrm{cl}_\theta F |\, F \in \Phi\}$. The filter base Φ *θ-converges* to its *θ-limit* x if for every closed neighbourhood H of x there is $F \in \Phi$ such that $F \subseteq H$. For any set S, we denote by $|S|$ the cardinality of S. For a family $\Phi \subseteq 2^X$, we denote by Φ^F the family of all finite unions of members of Φ. Let X be a topological space. We say that the points x, y are *T_2-separable* if they have open disjoint neighborhoods.

We say that a topological space X is *(countably) θ-regular* if every [5, 8] (countable) filter base in X with a θ-cluster point has a cluster point. A topological space X is said to be *point (countably) paracompact* [1] if for every open (countable) cover Ω of X and each $x \in X$ there is an open refinement Ω' of Ω such that Ω' is locally finite at x. Note that θ-regularity was originally defined by D. S. Janković [5] for a generalization of a Closed Graph Theorem of D. A. Rose. It should be pointed out that the term "a point paracompact space" due to J. M. Boyte [1] is completely different from another notion of *point-wise paracompact* which is also often called *metacompact* (defined as every open cover of the space has an open, point-finite refinement). In [8] the author proved that point paracompactness and θ-regularity coincide. The paper [8] is our starting point. We mention here its main definition and a the related characterization theorem. For the proof, the reader is referred to [8].

Definition 1.1. *Let m, n be cardinals. A space X is said to be (m, n)-cover regular if for each open cover Ω of X with $|\Omega| \leq m$ and each point $x \in X$ there is a closed neighbourhood of x, which can be covered by some*

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subcollection Ω' *of* Ω *such that* $|\Omega'| < n$. *A space, which is* (m,n)*-cover regular for every cardinal* m*, is called* (∞,n)*-cover regular.*

As it follows from author's result in [8], which we recapitulate here as Theorem 1.1 below, (∞,ω)-cover regular spaces are exactly the θ-regular spaces, (ω,ω)-cover regular spaces coincide with the class of countably θ-regular spaces and $(\infty,2)$-cover regular spaces are called cover regular in a paper[3] of Dowker and Strauss.

Theorem 1.1. *Let* X *be a topological space,* $m \geq \omega$ *a cardinal. The following statements are equivalent:*

(i) *For every open cover* Ω *of* X*,* $|\Omega| \leq m$*, and each* $x \in X$ *there is a closed neighborhood* G *of* x *such that* G *can be covered by a finite subfamily of* Ω*.*
(ii) *For every open cover* Ω *of* X*,* $|\Omega| \leq m$*, and each* $x \in X$ *there is an open refinement* Ω' *of* Ω *such that* Ω' *is locally finite at* x*.*
(iii) *For every filter base* Φ *in* X*,* $|\Phi| \leq m$*, having no cluster point and for each* $x \in X$ *there are* $F \in \Phi$ *and open disjoint sets* U*,* V *such that* $x \in U$ *and* $F \subseteq V$*.*
(iv) *Every filter base* Φ *in* X*,* $|\Phi| \leq m$*, with a* θ*-cluster point has a cluster point.*

Corollary 1.1. *The following statements are fulfilled:*

(i) *For any space,* (∞,ω)*-cover regularity,* θ*-regularity and point paracompactness are equivalent.*
(ii) *For any space,* (ω,ω)*-cover regularity, countable* θ*-regularity and countable point paracompactness are equivalent.*

Note that some preliminary results regarding the relationships between the classes of (m,n)-cover regular spaces of various cardinalities are also contained in [8]. More detailed study is in author's thesis [7].

2. Paracompactness

Note that in this paper we do not assume any additional separation axiom for paracompactness, so we say that a space X is *paracompact* if every open cover of X has an open locally finite refinement. Further, a topological space X is said to be *a-paracompact* [2] if every open cover of X has a locally finite (not necessarily open) refinement. And finally, a topological space is said to be *semiparacompact* [12] if every open cover of the space has a σ-locally finite open refinement.

In this section we derive a general method which allows to remove the restrictive presumption for a space being T_1 in some covering theorems concerning paracompactness. We will start with the following definition.

Definition 2.1. *Let* X *be a space and* $x, y \in X$ *be two points. We say that* y *absorbs* x *if every open neighbourhood of* y *contains* x*. A set* $Y \subseteq X$ *is said to be an absorbing set of* X *if every* $x \in X$ *is absorbed by any* $y \in Y$*. An absorbing set of* X *is called point-closed if its every point is closed in* X*.*

Note that the binary relation of absorbtion is always reflexive and transitive. In computer science motivated topology it is called *a preorder of specialization*.

Lemma 2.1. *Let X be a topological space which is T_0 and (∞, ω)-cover regular. Then X contains a point-closed absorbing set.*

Proof. For every $x, y \in X$ let $x \leqslant y$ (or, equivalently, $y \geqslant x$) if and only if y absorbs x. Clearly, the relation $\leqslant$ is transitive and reflexive. Since X is a T_0 space, $\leqslant$ is also antisymetric; hence it is an order on X. Let $M \subseteq X$ be a nonempty set which is a chain with respect to $\leqslant$. Pick any fixed $x \in M$ and take a closed neighbourhood G of x. For any $y \in M$, $y \geqslant x$ it follows, since y absorbs x, that $y \notin X \smallsetminus G$. Hence $y \in G$, which implies that the net $id_M(M, \geqslant)$ θ-converges to x. Since X is θ-regular by Corollary 1.1, $id_M(M, \geqslant)$ has a cluster point, say $z \in X$.

Let U be an open neighbourhood of z and take $x \in M$. It follows that there is some $y \in M$, $y \geqslant x$ such that $y \in U$. But y absorbs x, hence $x \in U$ as well. It follows that z absorbs every point of M which means that z is an upper bound of M. By Zorn's Lemma, every $x \in X$ is absorbed by some element of X which is maximal with respect to the order $\leqslant$. Let $Y = \{y | \, y \in X, y \text{ is maximal with respect to } \leqslant\}$. Obviously, Y is an absorbing subset of X. Let $y \in Y$, $x \in X$, $x \neq y$. It is not possible for x to absorb y since y is maximal. Then there exists an open neighbourhood V of x such that $y \notin V$. Therefore the set $\{y\}$ is closed in X and then Y is point-closed. □

Corollary 2.1. *Let X be an (∞, ω)-cover regular topological space. Then X contains an absorbing set which is T_1 in the induced topology.*

Proof. Suppose that X is not T_0 in general. For any $x, y \in X$ we put $x \sim y$ if and only if the points x, y absorb each other . The relation $\sim$ is a relation of equivalence on X; in fact, equivalent points have the same open neighbourhoods. Choose from each class of equivalence, associated with $\sim$, exactly one point $z \in X$ and denote by Z the set of all such points z. Clearly, $Z \subseteq X$ is an absorbing set of X which is a T_0 space in the induced topology. It is easy to show that Z is (∞, ω)-cover regular. Therefore, by the preceding lemma, the subspace Z containes a point-closed (with respect to the topology of Z) absorbing set, say $Y \subseteq Z$. Obviously, Y is T_1 in the induced topology. Now, let $x \in X$. There exists $z \in Z$ such that z absorbs x in X. Further, there is some $y \in Y$ such that y absorbs z in Z. Let $U \subseteq X$ be an open set in X such that $y \in U$. Then $U \cap Z$ is open in Z and then $z \in U \cap Z \subseteq U$. It follows that $x \in U$, which implies that y absorbs x. Hence, Y is the desired absorbing set. □

Lemma 2.2. *Let X be an (∞, ω)-cover regular topological space, Γ a locally finite collection of subsets of X. Then the following statements are fulfilled:*

(i) *The collection $\mathrm{cl}_\theta \Gamma = \{\mathrm{cl}_\theta G | G \in \Gamma\}$ is locally finite.*

(ii) *The collection Γ is θ-closure-preserving; for every $\Gamma' \subseteq \Gamma$ it follows that $\mathrm{cl}_\theta \bigcup_{G \in \Gamma'} G = \bigcup_{G \in \Gamma'} \mathrm{cl}_\theta G$.*

Proof. At first, let us show (i). Since Γ is locally finite, there is an open cover Ω of X such that every $U \in \Omega$ meets at most finitely many members of Γ. We may assume, without lose of generality, that Ω is directed. Since X is (∞, ω)-cover regular, every $x \in X$ has an open neighbourhood, say V, such that $\operatorname{cl} V \subseteq U$ for some $U \in \Omega$. Therefore, $\operatorname{cl} V$ meets also only a finite number of sets from Γ. Denote these sets $G_1, G_2, \dots, G_k$. Now, suppose that for some $G \in \Gamma$ we have $V \cap \operatorname{cl}_\theta G \neq \varnothing$. It follows that there is some $y \in V \cap \operatorname{cl}_\theta G$. Then $\operatorname{cl} V$ is a closed neighbourhood of y, which implies that $\operatorname{cl} V$ must meet G since $y \in \operatorname{cl}_\theta G$. It follows that $G \in \{G_1, G_2, \dots, G_k\}$. Hence, the collection $\operatorname{cl}_\theta \Gamma$ is locally finite.

Let us prove (ii). Let $\Gamma' \subseteq \Gamma$ be a nonempty set. Of course, $\bigcup_{G \in \Gamma'} \operatorname{cl}_\theta G \subseteq \operatorname{cl}_\theta \bigcup_{G \in \Gamma'} G$. Suppose that $x \in \operatorname{cl}_\theta \bigcup_{G \in \Gamma'} G$. From the previous part of the proof it follows that x has a closed neighbourhood, say H, which meets only a finite number of elements of Γ', say $G_1, G_2, \dots, G_n$. Then $x \in \operatorname{cl}_\theta G_i$ for some $i = 1, 2, \dots, n$. Therefore, it follows that $\operatorname{cl}_\theta \bigcup_{G \in \Gamma'} G = \bigcup_{G \in \Gamma'} \operatorname{cl}_\theta G$, which completes the proof.

□

Theorem 2.1. *For a space X, the following statements are equivalent:*

(i) *The space X is paracompact.*

(ii) *The space X is (∞, ω)-cover regular and has an absorbing set which is paracompact T_1 in the induced topology.*

(iii) *The space X is (∞, ω)-cover regular and has an absorbing set which is paracompact in the induced topology.*

Proof. (i) ⇒ (ii): Suppose (i). It follows from Corollary 1.1 that X is (∞, ω)-cover regular. Then, by Corollary 2.1, X has an absorbing set $Y \subseteq X$, which is a T_1 space in the induced topology. Obviously, every absorbing set of a paracompact space must be paracompact. Hence, we have (ii).

The implication (ii) ⇒ (iii) is clear.

(iii) ⇒ (i): Suppose (iii) and denote by Y the paracompact absorbing set of X. Let Ω be an open directed cover of X. There is an open cover Ω_1 of X such that for every $V \in \Omega_1$ there is some $U \in \Omega$ with $\operatorname{cl}_\theta V = \operatorname{cl} V \subseteq U$. It follows that $\operatorname{cl}_\theta \Omega_1$ refines Ω.

Let $\Phi = \{V \cap Y | \, V \in \Omega_1\}$. The collection Φ is an open cover in the induced topology of Y and, since Y is paracompact, it has a locally finite (in the space Y) refinement, say Φ_1. We show that Φ_1 is locally finite in X. Let $x \in X$. There is some $y \in Y$ which absorbs x. It follows that y has an open neighbourhood W such that the set $Y \cap W$ meets at most finitely many members of Φ_1. Clearly, $x \in W$ and since Φ_1 containes only subsets of Y, it follows that W meets only a finite number of elements of Φ_1 as well. Hence, Φ_1 is locally finite in X.

Since Φ_1 refines Ω_1, it follows that $\operatorname{cl}_\theta \Phi_1$ refines $\operatorname{cl}_\theta \Omega_1$ and hence it refines Ω^F as well. By Lemma 2.2 the collection $\operatorname{cl}_\theta \Phi_1$ is locally finite. It suffices to prove that $\operatorname{cl}_\theta \Phi_1$ covers X. Let $x \in X$. Since Y is an absorbing set, it follows that there exists $y \in Y$ which absorbs x. Then every closed neighbourhood of x meets y, which implies that $x \in \operatorname{cl}_\theta Y$. Hence $\operatorname{cl}_\theta Y = X$. Therefore, by Lemma 2.2, it follows $\bigcup_{S \in \Phi_1} \operatorname{cl}_\theta S = \operatorname{cl}_\theta \bigcup_{S \in \Phi_1} S = \operatorname{cl}_\theta Y = X$, which implies that $\operatorname{cl}_\theta \Phi_1$ covers X.

Since the directed open cover Ω has a locally finite refinement $\operatorname{cl}_\theta \Phi_1$, it has also a closed closure-preserving cover, whose interiors cover X. By a result of H. Junnila (Theorem 3.4, p. 379, [6]) X is paracompact, that is,

(i) holds. Note that an expansion of a locally finite cover $\mathrm{cl}_\theta \Phi_1$ to an open locally finite cover refining Ω is also possible (the reader may try it as an easy exercise by adjusting the original proof of E. Michael, see [9], where (∞, ω)-cover regular spaces should be used instead of regular spaces). □

The following theorem is an immediate consequence of the previous one.

Theorem 2.2. *Let $\mathcal{P}$ be some property of topological spaces, which satisfies the following conditions:*

(i) *If a space has $\mathcal{P}$, then every absorbing set $Y \subseteq X$ has $\mathcal{P}$ as a subspace.*
(ii) *The property of $T_1 + \mathcal{P}$ implies paracompactness.*

Then the property of (∞, ω)-cover regularity + $\mathcal{P}$ implies paracompactness. Moreover, if, in addition,

(iii) *paracompactness implies $\mathcal{P}$,*

then paracompactness is equivalent to (∞, ω)-cover regularity + $\mathcal{P}$.

Proof. Suppose that $\mathcal{P}$ satisfies the conditions (i) and (ii). Let X be an (∞, ω)-cover regular space which has the property $\mathcal{P}$. It follows from Corollary 2.1 that X has an absorbing set Y which is T_1 as a subspace of X and, by (i), has $\mathcal{P}$. Then, by (ii) the subspace Y is paracompact. It follows from Theorem 2.1 that X is paracompact as well.

Now, suppose that (iii) is fulfilled. Since paracompactness imply (∞, ω)-cover regularity by Corollary 1.1, the assertion is obvious. □

Corollary 2.2. *Let $\mathcal{Q}$ be a property of topological spaces, such that the following conditions are satisfied:*

(i) *If a space has $\mathcal{Q}$, then every absorbing set $Y \subseteq X$ has $\mathcal{Q}$ as a subspace.*
(ii) *The property of regularity + $\mathcal{Q}$ implies paracompactness.*

Then the property of $(\infty, 2)$-cover regularity + $\mathcal{Q}$ implies paracompactness.

Proof. We put $\mathcal{P} = (\infty, 2)$-cover regularity + $\mathcal{Q}$. Since every $(\infty, 2)$-cover regular T_1 space is regular, the property of $\mathrm{T}_1 + \mathcal{P}$ implies paracompactness. Because $(\infty, 2)$-cover regularity obviously satisfies (i), Theorem 2.2 completes the proof.

□

Remark 2.1. It is an unsolved problem (due to H. Junnila), whether every space such that every open directed cover of the space admits of an open σ-cushioned refinement is paracompact. Since this property, analogically as paracompactness, implies (∞, ω)-cover regularity, it follows that it suffices to solve the problem in T_1 spaces.

Remark 2.2. Obviously, Theorem 2.2 and Corollary 2.2 may be applied on several well-known covering theorems concerning paracompactness. Hence, for instance, in some cases regularity can be replaced by $(\infty, 2)$-cover regularity and the T_1-axiom by (∞, ω)-cover regularity.

Remark 2.3. Notice that paracompactness in the conditions (ii) and (iii) of Theorem 2.1 and the conditions (ii) of Theorem 2.2 and Corollary 2.2 may be replaced by some weaker properties; for example, by a-paracompactness or semiparacompactness.

3. Full normality

Similarly as for paracompactness in the previous section, we do assume the lower separation axioms such as T_i, where $i = 0, 1, 2$, for normality and full normality. So, we say that a space is *normal* if every two disjoint closed subsets of the space have open disjoint neighborhoods. Further, a space is said to be *fully normal* if every open cover of the space has an open star-refinement. It is well-known that fully normal spaces are paracompact. Conversely, it can be easily checked that if every open cover of the space admits of a closed locally finite refinement, the space is fully normal (see, e.g. [2], p. 346). Hence, for Hausdorff spaces, paracompactness and full normality are equivalent (see [2] or [11]). In the following we will show a little more; we will prove that full normality is exactly the same property as paracompactess plus normality. Paralelly, we will give some further characterizations of that property.

Lemma 3.1. *Any normal (∞, ω)-cover regular space X is $(\infty, 2)$-cover regular.*

Proof. Let X be normal and (∞, ω)-cover regular. At first, suppose that X is a T_0 space. Then, by Lemma 2.1, X has a point-closed absorbing set, say $Y \subseteq X$. Since X is normal, the points of Y are T_2-separable. Let Ω be an open cover of X and let $x \in X$. It follows that there is some $y \in Y$ absorbing x. Pick some $U \in \Omega$ such that $y \in U$. Then, for every $z \in Y \smallsetminus U$ we have an open neighbourhood $V(z)$ of z such that $y \notin \operatorname{cl} V(z)$. The collection $\{U\} \cup \{V(z) |\, z \in Y \smallsetminus U\}$ is an open cover of Y and therefore covers X. It follows that there are open neighbourhood W of y and $z_1, z_2, \dots, z_k \in Y \smallsetminus U$ such that $\operatorname{cl} W \subseteq U \cup \left[\bigcup_{i=1}^{k} V(z_i)\right]$. Let $S = W \smallsetminus \operatorname{cl} \bigcup_{i=1}^{k} V(z_i)$. Clearly, S is an open neighbourhood of y and then, since y absorbs x, also of x. On the other hand, one can easily check that $\operatorname{cl} S \subseteq U$, which implies that X is $(\infty, 2)$-cover regular.

Now, suppose that X is not T_0 in general. For every $x, y \in X$ we put $x \sim y$ if and only if the points x, y absorb each other. The relation $\sim$ is a relation of equivalence on X. Choose from each class of equivalence, associated with $\sim$, exactly one point $z \in X$ and denote by Z the set of all such points z. Clearly, $Z \subseteq X$ is an absorbing set of X which is a T_0 space with respect to the induced topology. Of course, Z is (∞, ω)-cover regular. We show that Z is normal. Let $A, B \subseteq Z$ be disjoint and closed with respect to the induced topology. There are open sets $U, V \subseteq X$ such that $Z \smallsetminus A = Z \cap U$, $Z \smallsetminus B = Z \cap V$. It follows that $Z = (Z \smallsetminus A) \cup (Z \smallsetminus B) \subseteq U \cup V$. Then, since Z is an absorbing set of X, it follows $U \cup V = X$. Since X is normal, there exist closed sets $G, H \subseteq X$ with $G \subseteq U$ and $H \subseteq V$ such that $G \cup H = X$. Let $P = Z \smallsetminus G$ and $Q = Z \smallsetminus H$. It follows that P, Q are disjoint, open in Z

and $A \subseteq P$, $B \subseteq Q$. Hence Z is normal. By the previous part of the proof it follows that Z is $(\infty, 2)$-cover regular.

Let Ω be an open cover of X. Since Ω covers Z, it follows that there exists a family Φ' which is a Z-open cover of Z such that the family $\overline{\Phi'} = \{\mathrm{cl}_Z F \mid F \in \Phi'\}$ refines Ω. For every $F \in \Phi'$ there exists some open $V(F) \subseteq X$ such that $F = V(F) \cap Z$. Obviously, the family $\Phi = \{V(F) \mid F \in \Phi'\}$ is an open cover of X. It suffices to show that the family $\overline{\Phi} = \{\mathrm{cl}\, V(F) \mid F \in \Phi'\}$ refines Ω. For $F \in \Phi'$, choose $U \in \Omega$ such that $\mathrm{cl}_Z F \subseteq U$. We prove that $\mathrm{cl}\, V(F) \subseteq U$. Let $x \in \mathrm{cl}\, V(F)$. There exists $z \in Z$ such that $z \sim x$. Let W be an open neighbourhood of z. Then $x \in W$, which implies that $W \cap V(F) \neq \varnothing$. Let $t \in W \cap V(F)$. There exists $s \in Z$ with $s \sim t$. It follows that $s \in W \cap V(F) \cap Z = (W \cap Z) \cap F \neq \varnothing$, and, hence, $z \in \mathrm{cl}_Z F \subseteq U$. Consequently, $x \in U$, which implies that $\mathrm{cl}\, V(F) \subseteq U$. But then it follows that the family $\overline{\Phi} = \{\mathrm{cl}\, V(F) \mid F \in \Phi'\}$ refines Ω. Therefore, X is $(\infty, 2)$-cover regular. □

Theorem 3.1. *For a space X, the following statements are equivalent:*

(i) *X is fully normal.*
(ii) *X is paracompact and normal.*
(iii) *X is paracompact and $(\infty, 2)$-cover regular.*

Proof. We metioned that fully normal spaces are paracompact. Hence (i) $\Rightarrow$ (ii). From Lemma 3.1 it follows that (ii) $\Rightarrow$ (iii). Finally, if X is paracompact and $(\infty, 2)$-cover regular, then every open cover of X admits of a locally finite closed refinement which implies (i). □

The following theorem is an analogy of Theorem 2.2 for full normality in place of paracompactness.

Theorem 3.2. *Let $\mathcal{R}$ be some property of topological spaces such that:*

(i) *If a space has $\mathcal{R}$, then every absorbing set $Y \subseteq X$ has $\mathcal{R}$ as a subspace.*
(ii) *The property $\mathcal{R}$ implies normality.*
(iii) *The property of $T_1 + \mathcal{R}$ implies paracompactness.*

Then the property of (∞, ω)-cover regularity + $\mathcal{R}$ implies full normality. Moreover, if, in addition,

(iv) *full normality implies $\mathcal{R}$,*

then full normality is equivalent to (∞, ω)-cover regularity + $\mathcal{R}$.

Proof. Suppose that $\mathcal{R}$ satisfies the conditions (i), (ii) and (iii). By Theorem 2.2 every space X, which is (∞, ω)-cover regular and $\mathcal{R}$, is paracompact. Hence, from (ii) and Theorem 3.1 it follows that X is fully normal.

Conversely, assume (iv). Since full normality implies that every open cover of the space has an open cushioned refinement, it follows that a fully normal space is $(\infty, 2)$-cover regular. The theorem now immediatelly follows. □

Corollary 3.1. *A space X is fully normal if and only if every open cover of X has a cushioned refinement.*

Proof. We denote by $\mathcal{R}$ the property 'every open cover of X has a cushioned refinement'. The conditions (i), (ii) and (iv) of Theorem 3.2 are obviously fulfilled. The condition (iii) is a well-known Michael's result; the reader is referred to Michael's paper [10] or to book [11]. Obviously, $\mathcal{R}$ implies (∞, ω)-cover regularity. Hence, Theorem 3.2 completes the proof. □

Corollary 3.2. *A space X is fully normal if and only if every open cover of X has a σ-cushioned open refinement.*

Proof. We let $\mathcal{R}$ = 'every open cover of X has a σ-cushioned open refinement'. Analogically as in the previous proof, one can check the conditions of Theorem 3.2. For (iii), see, for instance, [10] or [11]. Finally, it is clear that $\mathcal{R}$ implies (∞, ω)-cover regularity. Therefore, Theorem 3.2 completes the proof. □

In [6] H. Junnila proved that a space is paracompact if every interior-preserving open cover of the space has a closure-preserving closed refinement. In the light of Theorem 3.2 we may slightly improve this result to the following.

Corollary 3.3. *A space X is fully normal if and only if every interior-preserving open cover of X has a closure-preserving closed refinement.*

Proof. Only the sufficiency is non-trivial. Assume that X satisfies the condition stated in the assertion. Then, by Junnila's result, X is paracompact. Hence, every open cover of X admits of an open interior-preserving refinement (in fact, locally finite). Therefore, every open cover of X has a closure-preserving closed refinement. Now, Corollary 3.1 completes the prooof. □

We close the section by a consequence of Theorem 3.2, which slightly improves the result of H. H. Hung [4].

Theorem 3.3. *A space X is fully normal if and only if on every well-ordered open cover $(\Omega, \leqslant)$ there can be constructed functions ϑ^i where $i = 1, 2, \dots,$ into the family of all open sets, satisfying:*

(i) $\{\vartheta^j(V) | V \in \Omega, j = 1, 2, \dots\}$ *covers X,*

(ii) $\vartheta^i(U) \subseteq U,$

(iii) *"(iii)"*$\operatorname{cl} \bigcup \{\vartheta^i(V) | V \in \Omega, V < U\} \subseteq \bigcup \{V | V \in \Omega, V < U\}$

for every $U \in \Omega$ and $i = 1, 2, \dots$.

Proof. The necessity is clear. Let us prove the sufficiency. Assume that X satisfies the condition stated in the theorem. At first, we show that X is normal. Let $A, B \subseteq X$ be two disjoint closed sets. On the two well-ordered covers $\{X \setminus A, X \setminus B\}$, $\{X \setminus B, X \setminus A\}$ of X there are functions ϑ^i, η^i, $i = 1, 2, \dots,$ satisfying (i) – (iii). For every $i = 1, 2, \dots,$ let

$$W_i = \vartheta^i(X \setminus A) \setminus \operatorname{cl} \bigcup_{j=1}^{i-1} \eta^j(X \setminus B).$$

One can easily check that $B \subseteq \bigcup_{i \in \mathbb{N}} W_i$ and $A \cap (\operatorname{cl} \bigcup_{i \in \mathbb{N}} W_i) = \varnothing$. Therefore, X is normal.

By Hung's result [4] a T_1 space, having the property stated in the theorem, is paracompact. Hence, for the completness of the proof, by Theorem 3.2 it suffices to prove that X is (∞, ω)-cover regular. Let $(\Omega, \subseteq)$ be an open well-monotone cover of X and ϑ^i, $i = 1, 2, \dots$ the corresponding functions. We may assume, without loss of generality, that $X \notin \Omega$. Then Ω is unbounded with respect to $\subseteq$. For any $x \in X$ there exist some $V \in \Omega$ and $j \in \mathbb{N}$ such that $x \in \vartheta^j(V)$. By (iii), it follows that $\operatorname{cl} \vartheta^j(V) \subseteq U$, where U is the successor of V in $(\Omega, \subseteq)$. It follows that the θ-interiors of members of Ω cover X. Hence, by Theorem 1.1, X is (∞, ω)-cover regular. Theorem 3.2 completes the proof.

□

Remark 3.1. In fact, paracompactness in the conditions (iii) of Theorem 3.1 and (ii) of Theorem 3.2 may be replaced by some weaker property; for instance, by a-paracompactness or semiparacompactness. Hence, one can derive some other characterizations of full normality using the scheme of Theorem 3.2.

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