Conditioning with S-additive measures

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Denote by $S : [0,1]^2 \to [0,1]$ some commutative isotonic monoid, having 0 its neutral element and 1 its annihilator. S is called *triangular conorm, t-conorm* for short. For details on t-conorms see, e.g. [4].

Further, denote by $C : [0,1]^2 \to [0,1]$ some isotonic operator, having 1 its neutral element and 0 its annihilator. C is called *conjunctor*. More on conjunctors the reader can find in [2].

Let $([0,1], \mathcal{O}, \mu)$ be a measurable space with normed measure μ. The measure μ is said to be S-additive iff for each couple of disjoint sets $A, B \in \mathcal{O}$ the following yields:

$$\mu(A \cup B) = S(\mu(A), \mu(B)).$$

More on the S-additivity the reader can find, e.g., in [1, 3, 5].

Let us denote $\sigma(\mathcal{O} \times \mathcal{O})$ the least σ-algebra containing the system $\mathcal{O} \times \mathcal{O}$ and let $\nu : \sigma(\mathcal{O} \times \mathcal{O}) \to [0,1]$ be some two-dimensional S-additive measure. Then in an obvious way we can define marginal measures $\mu_1 : \mathcal{O} \to [0,1]$ and $\mu_2 : \mathcal{O} \to [0,1]$, which are also S-additive.

The well-known formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

can be generalized in the following way:

$$\mu(A|B) = \sup\{x \in [0,1]; C(x, \mu_2(B)) \leq \nu(A,B)\} \tag{1}$$

where μ_2 is a marginal measure of ν and C is some conjunctor. Iff moreover

$$\mu(A|B) = \mu_1(A) \tag{2}$$

then we say that event A is independent of event B. If equation (2) holds true for all couples A, B, then we say that the space $([0,1], \mathcal{O}, \mu_1)$ is independent of $([0,1], \mathcal{O}, \mu_2)$. Of course, replacing the conjunctor C by some other one may influence the independence.

Results

We say that a distibution function $F : [0,1] \to [0,1]$ generates some S-additive measure $\mu : \mathcal{O} \to [0,1]$, iff the measure of a semi-open interval $]a,b]$ is given by the formula:

$$\mu(]a,b]) = \sup\{x \in [0,1]; S(F(a), x) < F(b)\}.$$

We may speak also on two-dimensional distribution functions and their marginals and we will assume that all of them generate S-additive measures.

In fact, if $F : [0,1]^2 \to [0,1]$ is a two-dimensional distribution function, generating some S-additive measure, having $F_1,\ F_2$ as its marginals, then there exists some special conjunctor C (so-called S-copula) such that

$$F(x,y) = C(F_1(x), F_2(y)). \tag{3}$$

Obviously, if the space $([0,1], \mathcal{O}, \mu_1)$ is independent of $([0,1], \mathcal{O}, \mu_2)$, then the conjunctors used in formulas (1)and (3) must coincide. I.e., we will speak just of one conjunctor.

Theorem (a) Let $S = \max$. Then the space $([0,1], \mathcal{O}, \mu_1)$ is independent of $([0,1], \mathcal{O}, \mu_2)$ iff the conjunctor C is right-cancelative, i.e., for $y \neq 0$ we have $C(x_1, y) = C(x_2, y) \Rightarrow x_1 = x_2$.
(b) Let $S(x,y) = f^{-1}(\min\{1, f(x) + f(y)\})$ for some left-continuous strictly increasing function $f : [0,1] \to [0,1]$. Then the space $([0,1], \mathcal{O}, \mu_1)$ is independent of $([0,1], \mathcal{O}, \mu_2)$ iff the conjunctor C is given by $C(x,y) = f^{-1}(f(x) \cdot f(y))$.
(c) Let S be any other t-conorm. Then there are no independent spaces.

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