TOPOLOGY ON POWER SET
Vietoris topology

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Abstract

The aim of this remark is to show one possible way how to define a topology on a power set $\mathcal{P}^*(H)$ if the topology of the set H is given. In the paper we will recall the Vietoris topology defined in 1922 by an Austrian mathematician Leopold Vietoris (1891–2002) and show the way how to use it.

Assume that $(G,\cdot)$ is a groupoid and τ is a topology on G. If the binary operation $\cdot\colon G\times G \to G$ is continuous, the triple $(G,\cdot,\tau)$ is called a *topological groupoid.*

The continuity of binary operation "$\cdot$" means that for every $O\in\tau$ the set $\{(x,y)\in G\times G : x\cdot y\in O\}$ is open in $G\times G$ (endowed with ordinary product topology $\tau\times\tau$).

As the hyperoperation "$\cdot$" is a mapping from $H\times H$ to $\mathcal{P}^*(H)$ giving topologies on H and $\mathcal{P}^*(H)$ we can speak about the continuity of hyperoperation "$\cdot$" and define *topological hypergroupoid* or *topological hypergroup.* Unfortunately, if a topology is given on H, there is no standard straightforward way how to generate a topology on $\mathcal{P}^*(H)$.

First let us recall that a *base* $\mathcal{B}$ of a topological space (X,τ), i.e. a family of open sets such that any $O\in\tau$ is a union of sets from $\mathcal{B}$, has the following two properties [3]:

(B1) For any U_1, $U_2\in\mathcal{B}$ and every point $x\in U_1\cap U_2$ there exists a $U_3\in\mathcal{B}$ such that $x\in U_3\subset U_1\cap U_2$.

(B2) For every $x\in X$ there exists a $U\in\mathcal{B}$ such that $x\in U$.

Conversely, any system $\mathcal{B}$ of sets fulfilling (B1) and (B2) determines a unique topology τ on X such that this system is a base of τ.

Further, a family of open sets $\mathcal{S}$ is called a *subbase* of a topological space (X,τ) if the family of all finite intersections $U_1\cap U_2\cap\cdots\cap U_k$, where $U_i\in\mathcal{S}$ for $i=1,2,\ldots,k$, is a base for (X,τ).

Any family $\mathcal{S}$ of subsets of X such that its union is X serves as a subbase of a unique topology on X.

Recall as well the basic terms and definitions of hyperstructure theory which will be used in this paper—see [1, 2, 4].

Let H be a nonempty set and $\mathcal{P}^*(H)$ the set of all nonempty subsets of H. Then a *hypergroupoid* is a pair $(H,\cdot)$, where $\cdot\colon H\times H\to\mathcal{P}^*(H)$ is a binary *hyperoperation* operation on the set H. If $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ for all $a,b,c\in H$, (associativity axiom), then $(H,\cdot)$ is called a *semihypergroup.* A semihypergroup $(H,\cdot)$ is said to be a *hypergroup* if the following reproduction axiom is satisfied: $a\cdot H=H=H\cdot a$ for all a $\in H$.

For $A,B\subseteq H$, $A\neq\emptyset\neq B$ we define as usual $A\cdot B=\bigcup\{a\cdot b;\, a\in A, b\in B\}$. The notations $a\cdot A$ and $A\cdot a$ are used for $\{a\}A$ and $A\{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

In 1922 an Austrian mathematician L. Vietoris (1891—2002) [6], see also [3, p. 162], defined on the collection of all nonempty closed subsets $\mathbf{CL(S)}$ of a topological space (S,τ) a topology as follows: For each finite collection $U_1, \dots U_n \in \tau$, let $\langle U_1, \dots U_n \rangle$ denote the set of all A in $\mathbf{CL(S)}$ such that $A \subset \bigcup_{i=1}^{n}$ and $A \cap U_i \neq \emptyset$ for each $i = 1, 2, \dots, n$; the sets $\langle U_1, \dots U_n \rangle$ form the base of a topology on $\mathbf{CL(S)}$.

In the case of compact connected metric space S this topology coincides with the topology induced by the so-called "Hausdorff metric". Recently, this metric has again become important, for example in fractal image compression.

His approach can be applied to the collection of all nonempty subsets of (S, τ) as follows:

Let (H, τ) be a topological space. For any $U_1, \dots U_k$, $k \in \mathbb{N}$, $U_i \in \tau$ for $i = 1, \dots, k$, let us denote

$$\mathcal{V}(U_1, \dots U_k) = \Big\{ B \in \mathcal{P}^*(H) : B \subset \bigcup_{i=1}^{k} U_i \text{ and } B \approx U_i \text{ for } i = 1, \dots, k \Big\}.$$

The family $\mathcal{B}$ of all sets $\mathcal{V}(U_1, \dots U_k)$ forms a base of a topological space $(\mathcal{P}^*(H), \tau_{\mathcal{V}})$. The topology $\tau_{\mathcal{V}}$ is called a *Vietoris topology* on $\mathcal{P}^*(H)$.

Assume now that the triple $(G, \cdot, \tau)$ is a *topological groupoid*. As was mentioned above the continuity of "$\cdot$" means that for every $O \in \tau$ the set $\{(x, y) \in G \times G : x \cdot y \in O\}$ is open in $G \times G$. If we consider a binary operation than the condition given on a mapping to be continuous is that full inverse image of an arbitrary open set is an open set. As the hyperoperation "$\cdot$" is a mapping from $H \times H$ to $\mathcal{P}^*(H)$ giving topologies on H and $\mathcal{P}^*(H)$ we can speak about the continuity of "$\cdot$". If we consider a hyperoperation (so the product of two elements is a set resp. subset of H) there are two ways how to define a continuous mapping:

I. First—the product, i.e. the whole set, has to be a subset of an opened set O.

II. Second—the product has to have a nonempty intersection with an opened set O.

As we mentioned at the beginning of this paper if a topology is given on H, there is no standard straightforward way how to generate a topology on $\mathcal{P}^*(H)$, but from the previous it is clear that the Vietoris topology is one of the possible ways how to define a topology on $\mathcal{P}^*(H)$ if a topology on H is given. See [5] for more details.

References

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