A note on fuzzy preference structures

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1 Motivation Example

Grabisch and Roubens consider the problem of the evaluation of trainees learning to drive military vehicles. The instructors evalueted the trainees according to 4 criteria:

C1. Firing precision: The percentage of success during the exercise is computed.

C2. Target detection rapidity: The elapsed time between the appearance of the target and the detection is measured in tu (time unit).

C3. Driving: In order to go from one point to another, the trainee has to choose a suitable trajectory, or to follow a given one as precisely as possible. A qualitive score is given by the instructor, ranging from A (excelent) to E (hopeless).

C4. Communication: The trainee is supposed to belong to some unit, and thus he should understand and obey orders, and also report actions. As for the driving criterion, a qualitative score is given by the instructor, ranging from A (perfect) to E (incontrollable).

Table 1.1: Performances of the different trainees.

name	*precision (%)*	*rapidity (tu)*	*driving*	*communication*
Arthur	90	2	B	D
Lancelot	80	4	B	B
Yvain	95	5	C	A
Perceval	60	6	B	B
Erec	65	2	C	B

Instructor's comments:

C.1 (precision): over 90% of success is perfect, below 50% is totally unacceptable.

C.2 (rapidity): below 2 tu is perfect, over 10 tu is totally unacceptable.

C.3 and C.4: these criteria are already expressed in the form of an equidistant numerical score.

Table 1.2: Scores on the different criteria

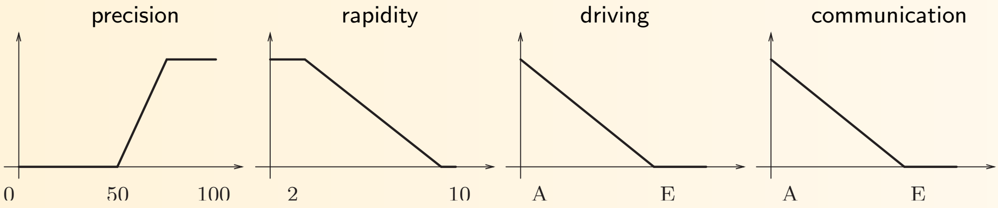

Table 1.3: Numerical scores on criteria.

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750

Table 1.4: Ranking of the five trainees.

name	*class*	*rank in the class*
Arthur	bad	2
Lancelot	good	1
Yvain	good	2
Perceval	bad	1
Erec	average	1

In [Grabisch and Roubens] an approach is taken, where the global ranking is represented as Choquet integral, and we have to learn the measure. The condition for learning is either;

1. Approach by the minimization of the quadratic error,

or

2. Approach based on constraint satisfaction.

Table 1.5: Mapping from class and rank to $[0,1]$.

class	*interval for the global score*
good	[0.75, 1.0]
average	[0.4, 0.75]
bad	[0.0, 0.4]

Our approach is based on conenction between fuzzy and annotated logic programs and an inductive logic programming method for learning rules of annotated programs.

Table 1.6: Numerical data on criteria and global performance.

name	*precision*	*rapidity*	*driving*	*communication*	global 1st
Arthur	1.000	1.000	0.750	0.250	0.133
Lancelot	0.750	0.750	0.750	0.750	0.917
Yvain	1.000	0.625	0.500	1.000	0.833
Perceval	0.250	0.500	0.750	0.750	0.276
Erec	0.375	1.000	0.500	0.750	0.575

name	*precision*	*rapidity*	*driving*	*communication*	global 2nd
Arthur	1.000	1.000	0.750	0.250	0.3
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.7
Perceval	0.250	0.500	0.750	0.750	0.35
Erec	0.375	1.000	0.500	0.750	0.5

Table 1.7: Linear ranking of the five trainees

name	*global rank*
Arthur	0.125
Lancelot	0.875
Yvain	0.75
Perceval	0.375
Erec	0.625

Table 1.8: Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*	*global rank*
Arthur	1.000	1.000	0.750	0.250	0.125
Lancelot	0.750	0.750	0.750	0.750	0.875
Yvain	1.000	0.625	0.500	1.000	0.75
Perceval	0.250	0.500	0.750	0.750	0.375
Erec	0.375	1.000	0.500	0.750	0.625

2 Introduction to preference structures and fuzzy preference structures

The preference structure is a basic step of preference modeling. Given two alternatives, decision maker defines three binary relation-preference, indifference and incomparability.

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS) a decision-maker makes three decission for any par (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision define a triplet P, I, J of a crisp binary relations on $\boldsymbol{A}$:

1) a is prefered to $b \Leftrightarrow (a, b) \in P$ (strict preference).
2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).
3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that (ps1) I is reflexive, P and J are irreflexive.

(ps2) P is asymmetric, I and J are symmetric.

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.

(ps4) $P \cup I \cup J \cup P^t = A \times A$ where $P^t(x,y) = P(y,x)$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the large preference relation. The relation R can be interpreted as

$$(a,b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J$$

where $coR(a,b) = 1 - R(a,b)$ and

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t).$$

Let (T,S,N) be De Morgan triplet. A fuzzy preference structure (FPS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary fuzzy relations on $\boldsymbol{A}$ such that

(f1) I is reflexive, P and J are irreflexive. $I(a,a) = 1, P(a,a) = J(a,a) = 0$

(f2) P is T-asymmetrical, I and J are symmetrical. $T(P(a,b), P(b,a)) = 0$

(f3) $T(P,I) = T(P,J) = T(I,J) = 0$. for all pair of alternatives

(f4) $(\forall (a,b) \in A^2) S(P, P^t, I, J) = 1$ or $N(S(P,I)) = S(P^t, J)$ or another completeness conditions.

3 Preference structures and fuzzy preference structures and their applications

R_P	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	1	0
L	0	1	1	1	0
P	0	0	0	1	0
Y	1	1	1	1	1

R_R	A	E	L	P	Y
A	1	1	1	1	1
E	1	1	1	1	1
L	0	0	1	1	1
P	0	0	0	1	0
Y	0	0	0	1	1

$R_P = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],$
$[A,Y],[Y,A],[A,L],[A,E],[A,P],[Y,L],$
$[Y,E],[Y,P],[L,E],[L,P],[E,P]\}$

$R_R = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],$
$[A,E],[E,A],[A,L],[A,Y],[A,P],[E,L],$
$[E,Y],[E,P],[L,Y],[L,P],[Y,P]\}$

R_D	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	0	1
L	1	1	1	1	1
P	1	1	1	1	1
Y	0	1	0	0	1

R_C	A	E	L	P	Y
A	1	0	0	0	0
E	1	1	1	1	0
L	1	1	1	1	0
P	1	1	1	1	0
Y	1	1	1	1	1

$R_D = \{[A, A], [E, E], [L, L], [P, P], [Y, Y], [A, L], [L, A], [A, P], [P, A], [L, P], [P, L], [A, E], [A, Y], [L, E], [L, Y], [P, E], [P, Y], [E, Y], [Y, E]\}$

$R_C = \{[A, A], [E, E], [L, L], [P, P], [Y, Y], [Y, L], [Y, P], [Y, E], [Y, A], [L, P], [P, L], [L, E], [E, L], [P, E], [E, P], [L, A], [P, A], [E, A]\}$

And we are able to construct large preference relation R_I which is derived from instructor's global ordering, too:

R_I	A	E	L	P	Y
A	1	0	0	1	0
E	1	1	0	1	0
L	1	1	1	1	1
P	1	0	0	1	0
Y	1	1	1	1	1

$$R_I = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[L,Y],[Y,L],\ [L,E],[L,A],[L,P],[Y,E],[Y,A],\\ [Y,P],[E,A],[E,P],[A,P],[P,A]\}$$

The relation R_I is a quasi order set. For global evaluation we will modify this quasi ordering to linear ordering. First, we need order the criteria.

The first idea is: we can pairwise compare the relations R_P, R_R, R_D and R_C with respect to relation R_I by the following rule:

$$X > Y \iff \frac{|R_X \cap R_I|}{|R_X \triangle R_I|} > \frac{|R_Y \cap R_I|}{|R_Y \triangle R_I|}, \quad (1)$$

where $X, Y \in \{P, R, D, C\}$. The idea is: the more R_X is similar to R_I, the more important criterion are X is. This method gives the following ordering of criteria:

$$\text{communication} > \text{precision} > \text{rapidity} > \text{driving}.$$

P_P	A	E	L	P	Y
A	0	1	1	1	0
E	0	0	0	1	0
L	0	1	0	1	0
P	0	0	0	0	0
Y	0	1	1	1	0

$P_P = \{[A,L],[A,E],[A,P],[Y,L],[Y,E],$
$[Y,P],[L,E],[L,P],[E,P]\}$

P_R	A	E	L	P	Y
A	0	0	1	1	1
E	0	0	1	1	1
L	0	0	0	1	1
P	0	0	0	0	0
Y	0	0	0	1	0

$P_R = \{[A,L],[A,Y],[A,P],[E,L],[E,Y],$
$[E,P],[L,Y],[L,P],[Y,P]\}$

P_D	A	E	L	P	Y
A	0	1	0	0	1
E	0	0	0	0	0
L	0	1	0	0	1
P	0	1	0	0	1
Y	0	0	0	0	0

$P_D = \{[A,E],[A,Y],[L,E],[L,Y],[P,E],$
$[P,Y]\}$

P_C	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	0	0
L	1	0	0	0	0
P	1	0	0	0	0
Y	1	1	1	1	0

$P_C = \{[Y,L],[Y,P],[Y,E],[Y,A],[L,A],$
$[P,A],[E,A]\}$

P_I	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	1	0
L	1	1	0	1	0
P	0	0	0	0	0
Y	1	1	0	1	0

$$P_I = \{[L,E],[L,A],[L,P],[Y,E],[Y,A],[Y,P],[E,A],[E,P]\}$$

Fuzzification

The value of fuzzy preference in precision (FP_P) for Arthur and Erec, we compute from Table 3 as $FP_P(A, E) = \max\{x_1^A - x_1^E, 0\}$, where x_1^A and x_1^E are Arthur's and Erec's precision score in Table 3, etc.

FP_P	A	E	L	P	Y
A	0	0.625	0.25	0.75	0
E	0	0	0	0.125	0
L	0	0.375	0	0.5	0
P	0	0	0	0	0
Y	0	0.625	0.25	0.75	0

FP_R	A	E	L	P	Y
A	0	0	0.25	0.5	0.375
E	0	0	0.25	0.5	0.375
L	0	0	0	0.25	0.125
P	0	0	0	0	0
Y	0	0	0	0.125	0

FP_D	A	E	L	P	Y
A	0	0.25	0	0	0.25
E	0	0	0	0	0
L	0	0.25	0	0	0.25
P	0	0.25	0	0	0.25
Y	0	0	0	0	0

FP_C	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0	0
L	0.5	0	0	0	0
P	0.5	0	0	0	0
Y	0.75	0.25	0.25	0.25	0

FP_I	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0.25	0
L	0.75	0.25	0	0.5	0.125
P	0.125	0	0	0	0
Y	0.75	0.25	0	0.5	0

communication $\succ$ precision $=$ driving $\succ$ rapidity.

Table 3.1: Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750
Bruno	0.400	0.750	0.600	0.750

Simple deduction.

The final ordering of trainees is:

$$\text{Yvain} > \text{Lancelot} > \text{Bruno} > \text{Erec} > \text{Perceval} > \text{Arthur}.$$